Existence of renormalized solutions to nonlinear elliptic equations with two lower order terms and measure data

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Abstract In this paper we prove the existence of a renormalized solution to a class of nonlinear elliptic problems whose prototype is

$$(P) \qquad \begin{cases} -\triangle_p u - \mathrm{div}\,(c(x)|u|^\gamma) + b(x)|\nabla u|^\lambda = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\triangle_p$ is the so called $p-$Laplace operator, $1 < p < N$, μ is a Radon measure with bounded variation on Ω, $0 \leq \gamma \leq p-1$, $0 \leq \lambda \leq p-1$, $|c|$ and b belong to the Lorentz spaces $L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$ and $L^{N,1}(\Omega)$ respectively. In particular we prove the existence under the assumption that $\gamma = \lambda = p-1$, $|c|$ belongs to the Lorentz space $L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r < +\infty$ and $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$ is small enough.

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1 Introduction

In this paper we consider nonlinear elliptic problems whose prototype is

$$\begin{cases} -\triangle_p u - \mathrm{div}\,(c(x)|u|^\gamma) + b(x)|\nabla u|^\lambda = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\triangle_p$ is the so called $p-$Laplace operator, p is a real number such that $1 < p < N$, μ is a Radon measure with bounded variation on Ω,

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$0 \leq \gamma \leq p-1$, $0 \leq \lambda \leq p-1$, $|c|$ and b belong to the Lorentz spaces $L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$ and $L^{N,1}(\Omega)$, respectively.

We are interested in existence results for renormalized solutions to (1.1).

We have proved such an existence result in [GM], when μ is a Radon measure with bounded variation on Ω, $\gamma = \lambda = p-1$, $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $r < +\infty$, is large and $\|b\|_{L^{N,1}(\Omega)}$ is small enough; the existence of a renormalized solution is also obtained, without assumption on the smallness of the norms of the coefficients, when γ or λ are less than $p-1$.

In the present paper we investigate the counterpart of the existence result given in [GM], that is we prove the existence of a renormalized solution when μ is a Radon measure with bounded variation on Ω, $\gamma = p-1$, $\lambda = p-1$, $\|b\|_{L^{N,1}(\Omega)}$ is large and $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $r < +\infty$ is small. The case $\gamma < p-1$ (and $\lambda \leq p-1$) is also studied.

The main features of (1.1) are both the fact that the operator has two lower order terms, which produce a lack of coercivity, and the right-hand side which is a measure.

Let us assume that the operator has not lower order terms, i.e. $b = c = 0$; in this case the difficulties in studying problem (1.1) are due only to the right-hand side μ.

Simple examples (the Laplace operator in a ball, i.e. $p = 2$, $b = 0$, $c = 0$, and μ the Dirac mass in the center) show that, in general, the solution of (1.1) does not belong to the space $W^{1,1}_{loc}(\Omega)$. Thus it is necessary to change the classical framework of Sobolev spaces in order to prove existence results.

In the linear case, i.e. $p = 2$, Stampacchia introduced a notion of solution of problem (1.1) defined by "duality"([St]) for which he proved the existence and the uniqueness. He also proved that such a solution satisfies the equation in distributional sense and it belongs to $W^{1,q}_0(\Omega)$ for every $q < N/(N-1)$. Unfortunately, Stampacchia's arguments cannot be extended to the nonlinear case except in the case where $p = 2$ as shown in [M2].

The first attempt in studying the nonlinear case was done by Boccardo and Gallouet ([BG1], [BG2]), who proved, under the assumption $p > 2 - \frac{1}{N}$, the existence of a solution which satisfies the equation in the distributional sense and which belongs to $W^{1,q}_0(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$. Let us explicitly remark that the assumption on p implies that $\frac{N(p-1)}{N-1} > 1$.

The next step thus consisted to find an "extra condition" on the distributional solutions of (1.1) in order to prove both existence and uniqueness results. This is done by introducing two equivalent notions of solution: the notion of entropy solution in [BBGGPV], [BGO] and the notion of the renormalized solution in [LM], [M1]. These settings were, however, limited to the case of measure in $L^1(\Omega)$ or in $L^1(\Omega) + W^{-1,p'}(\Omega)$. The case of a general measure with bounded total variation was studied in [DMOP], where the notion of renormalized solution has been extended to this case and an existence result is proved.

The effect of the two terms $b(x)|\nabla u|^{\lambda}$ and $-\text{div}(c(x)|u|^{\gamma})$ is a lack of coercivity of the operator.

In the linear case, i.e. $p = 2$, $\gamma = \lambda = 1$, Stampacchia proved the existence and the uniqueness of a "duality" solution , if 0 is not in the spectrum of the operator. Such condition is verified if, for example, $\|c\|_{L^{\frac{N}{p-1}}(\Omega)}$ or $\|b\|_{L^N(\Omega)}$ is small enough. The case of a nonlinear operator was studied in [D], where a term $b(x)|\nabla u|^\lambda$ is considered, and in [DPo1], where both terms $-\text{div}\,(c(x)|u|^\gamma)$ and $b(x)|\nabla u|^\lambda$ are considered; in these papers the existence of a solution which satisfies the equation in distributional sense is proved.

The effects of both the right-hand side a measure and the lower order term $b(x)|\nabla u|^\lambda$ were studied in [BMMP3], where the existence of a renormalized solution is proved.

Existence results for entropy solutions are proved by Boccardo in [B] when the operator has a lower order term of the type $-\text{div}(c(x)u)$. Moreover, in the nonlinear case when the operator has a lower order term of the type $-\text{div}(c(x)|u|^\gamma)$ and the right-hand side μ belongs to $L^1(\Omega)$, the existence of a renormalized solution is proved in [BGu1], [BGu2].

Finally let us explain the restriction $p < N$. If p is greater than the dimension N of the ambient space, then, by the Sobolev embedding theorem and duality arguments, the space of measures with bounded variation on Ω is a subset of $W^{-1,p'}(\Omega)$, so that the existence of solutions in $W_0^{1,p}(\Omega)$ was proved by Stampacchia in the linear case, i.e. $p = 2$, $\gamma = \lambda = 1$ (see also [Dr]) and by [DPo2] (see also [G2] for a different proof).

Uniqueness results for renormalized solutions can be found in [BMMP2], when the datum μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ and the operator has a lower order term of the type $b(x)|\nabla u|^\lambda$ (see [BMMP4] for the case where μ belongs to $W^{-1,p'}(\Omega)$) and in [BGu1], [BGu2] when μ belongs to $L^1(\Omega)$ and a lower order term of the type $-\text{div}(c(x)|u|^\gamma)$ is considered (see also [G1] for further uniqueness results).

In the present paper we consider operators where both the two lower order terms $-\text{div}\,(c(x)|u|^\gamma)$ and $b(x)|\nabla u|^\lambda$ appear without any coerciveness assumption on the operator.

Our main result is Theorem 3.1 in Section 2.2. It is an existence result for a class of nonlinear elliptic problems whose model is the problem (1.1). In the model case such a theorem states that at least a renormalized solution exists if one of the following condition holds true

1) $\gamma = p-1$, $c \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ and $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$ is small enough;

2) $\gamma < p-1$ and $c \in L^{\frac{N}{p-1},\infty}(\Omega)$.

The proof of such a result is obtained in various steps. The first difficulty is to obtain some a priori estimate for $|\nabla u|^{p-1}$. By adapting a technique used in [G2] (cf. [B]), this is done by decomposing $|\nabla u|^{p-1}$ in two terms

$$\begin{aligned} |\nabla u|^{p-1} &= \chi_{\{|u|\le m_1\}}|\nabla u|^{p-1} + \chi_{\{|u|>m_1\}}|\nabla u|^{p-1} \\ &= |\nabla T_{m_1}(u)|^{p-1} + |\nabla S_{m_1}(u)|^{p-1}, \end{aligned}$$

where $S_{m_1}(u) = u - T_{m_1}(u)$ is the "remainder" of the truncation $T_{m_1}(u)$ and m_1 is a value suitably chosen. Then we firstly prove an priori estimate for $|\nabla S_{m_1}(u)|^{p-1}$; in this step we use a generalization, proved in [BMMP3], of a result of [BBGGPV], which says that if v is a function such that $T_k(v) \in W_0^{1,p}(\Omega)$ and if $\|\nabla T_k(v)\|^p_{(L^p(\Omega))^N} \le kM + L$, for every $k > 0$, then $\||\nabla v|^{p-1}\|_{L^{N',\infty}(\Omega)} \le c$, where c depends on M, L and Ω. Then we prove that m_1 is uniformly bounded by a constant which depends only on the data c, b, μ and Ω and this gives the desired a priori estimate of $|\nabla u|^{p-1}$. Finally we use a stability result, proved in [GM] for equations whose prototype is (1.1) with $b = 0$, which is an extension of the stability result proved in [DMOP] (see also [MP]). We also recall that in [GM] we prove the counterpart of Theorem 3.1, that is we prove the existence of a renormalized solution when μ is a Radon measure with bounded variation on Ω, $\gamma = \lambda = p-1$, $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $r < +\infty$, is large and $\|b\|_{L^{N,1}(\Omega)}$ is small enough. It is worth noting that the method used in the present paper to obtain the a priori estimates seems not allow to deal with the case $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$ large ($r < +\infty$) and $\|b\|_{L^{N,1}(\Omega)}$ small enough while it seems that the one performed in [GM] is not suitable to the case $\|b\|_{L^{N,1}(\Omega)}$ large and $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$ small ($r < \infty$). We explicitly remark that the results proved in the present paper and those proved in [GM] imply the existence of a renormalized solution to the model problem (1.1) under the assumption that the norm of the coefficient c or the norm of the coefficient b is small enough.

2 Notation and definition of renormalized solution

2.1 Notation and definitions

In this section we recall some well-known results about the decomposition of measures (cf. [DMOP]) and a few properties of Lorentz spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 2$. Let p and p' be real numbers such that $1 < p < N$ and p' the Hølder conjugate exponent of p, i.e. $1/p + 1/p' = 1$.

We denote by $M_b(\Omega)$ the space of Radon measures on Ω with bounded total variation and by $C_b^0(\Omega)$ the space of bounded, continuous functions on Ω. Moreover μ^+ and μ^- denote the positive and the negative parts of the measure μ, respectively. We say that a sequence $\{\mu_\varepsilon\}$ of measures in $M_b(\Omega)$ converges in the narrow topology to a measure μ in $M_b(\Omega)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \, d\mu_\varepsilon = \int_\Omega \varphi \, d\mu,$$

for every $\varphi \in C_b^0(\Omega)$.

We denote by $\mathrm{cap}_p(B, \Omega)$ the standard capacity defined from $W_0^{1,p}(\Omega)$ of a Borel set B and we define $M_0(\Omega)$ as the set of the measures μ in $M_b(\Omega)$ which are absolutely continuous with respect to the p-capacity, i.e. which satisfy $\mu(B) = 0$ for every Borel set

$B \subseteq \Omega$ such that $\text{cap}_p(B,\Omega) = 0$. We define $M_s(\Omega)$ as the set of all the measures μ in $M_b(\Omega)$ which are singular with respect to the p-capacity, i.e. which are concentrated in a set $E \subset \Omega$ such that $\text{cap}_p(E,\Omega) = 0$.

The following result allows to split every measure in $M_b(\Omega)$ with respect to the p-capacity ([FST], Lemma 2.1 and [BGO], Theorem 2.1).

Proposition 2.1 *For every measure μ in $M_b(\Omega)$ there exists an unique pair of measures (μ_0, μ_s), with $\mu_0 \in M_0(\Omega)$ and $\mu_s \in M_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$. Moreover for any μ_0 belongs to $M_0(\Omega)$, there exists f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$ such that*

$$\mu_0 = f - \text{div}(g),$$

in the sense of distributions.

The measures μ_0 and μ_s will be called the absolutely continuous part and the singular part of μ with respect to the p-capacity.

We also recall that every function $v \in W_0^{1,p}(\Omega)$ is measurable with respect to μ_0 and belongs to $L^\infty(\Omega, \mu_0)$. If v further belongs to $L^\infty(\Omega)$, one has

$$\int_\Omega v\, d\mu_0 = \int_\Omega f v + \int_\Omega g \nabla v, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

(see, e.g., [DMOP], Proposition 2.7)

Combining the previous result and the Hahn decomposition Theorem, we get the following result

Proposition 2.2 *Every measure μ in $M_b(\Omega)$ can be decomposed as follows*

$$\mu = \mu_0 + \mu_s = f - \text{div}(g) + \mu_s^+ - \mu_s^-,$$

where μ_0 is a measure in $M_0(\Omega)$, hence can be written as $f - \text{div}(g)$, with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, and where μ_s^+ and μ_s^- (the positive and the negative parts of μ_s) are two nonnegative measures in $M_s(\Omega)$, which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity.

We recall now the definition and a few properties of Lorentz spaces, which we will use in the following. For references about Lorentz spaces see, for example, [Lo, O].

Let us denote by f^* the decreasing rearrangement of f, i.e. the decreasing function defined by

$$f^*(t) = \inf\{s \geq 0 \,:\, \text{meas}\left\{x \in \Omega \,:\, |f(x)| > s\right\} < t\}, \quad t \in [0, |\Omega|].$$

For references about rearrangements see, for example, [CR, K].

Moreover for $1 < q < \infty$ and $1 < s \leq \infty$, denote

$$\|f\|_{L^{q,s}(\Omega)} = \begin{cases} \left(\int_0^{|\Omega|} [f^*(t)t^{\frac{1}{q}}]^s \frac{dt}{t}\right)^{1/s}, & \text{if } s < \infty, \\ \sup_{t>0} t\left[\text{meas}\left\{x \in \Omega : |f(x)| > t\right\}\right]^{1/r}, & \text{if } s = \infty, \end{cases} \tag{2.1}$$

The Lorentz space $L^{q,s}(\Omega)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,s}(\Omega)} < +\infty, \tag{2.2}$$

endowed with the norm defined by (2.1).

They are "intermediate spaces" between the Lebesgue spaces, in the sense that, for every $1 < s < r < \infty$, we have

$$L^{r,1}(\Omega) \subset L^{r,r}(\Omega) = L^r(\Omega) \subset L^{r,\infty}(\Omega) \subset L^{s,1}(\Omega). \tag{2.3}$$

The space $L^{r,\infty}(\Omega)$ is the dual space of $L^{r',1}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, and one has the generalized Hølder inequality

$$\begin{cases} \forall f \in L^{r,\infty}(\Omega),\ \forall g \in L^{r',1}(\Omega), \\ \int_\Omega |fg| \leq \|f\|_{L^{r,\infty}(\Omega)} \|g\|_{L^{r',1}(\Omega)}. \end{cases} \tag{2.4}$$

More generally, if $1 < p < \infty$ and $1 \leq q \leq \infty$, we get

$$\begin{cases} \forall f \in L^{p_1,q_1}(\Omega),\ \forall g \in L^{p_2,q_2}(\Omega), \\ \|fg\|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{p_1,q_1}(\Omega)} \|g\|_{L^{p_2,q_2}(\Omega)}, \\ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \end{cases} \tag{2.5}$$

Improvements of the classical Sobolev inequalities involving Lorentz spaces are proved, for example, in [ALT]. In the present paper we will only use the following generalized Sobolev inequality: there exists a positive constant $S_{N,p}$ depending only on p and N such that

$$\|v\|_{L^{p^*,p}(\Omega)} \leq S_{N,p} \|v\|_{W_0^{1,p}(\Omega)}, \tag{2.6}$$

for every $v \in W_0^{1,p}(\Omega)$.

2.2 Definition of renormalized solution

In the present paper we consider a nonlinear elliptic problem which can formally be written as

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)+K(x,u))+H(x,u,\nabla u)+G(x,u)=\mu-\mathrm{div}(F) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Here $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ and $K:\Omega\times\mathbb{R}\to\mathbb{R}^N$ are Carathéodory functions satisfying

$$a(x,s,\xi)\xi\geq\alpha|\xi|^p, \quad \alpha>0, \tag{2.8}$$

$$|a(x,s,\xi)|\leq c\left[|\xi|^{p-1}+|s|^{p-1}+a_0(x)\right], \quad a_0(x)\in L^{p'}(\Omega), \quad c>0, \tag{2.9}$$

$$\big(a(x,s,\xi)-a(x,s,\eta),\xi-\eta\big)>0, \quad \xi\neq\eta, \tag{2.10}$$

$$\begin{cases} |K(x,s)|\leq c_0(x)|s|^\gamma+c_1(x), \\ \\ 0\leq\gamma\leq p-1, \quad c_0\in L^{\frac{N}{p-1},r}(\Omega), \quad \frac{N}{p-1}\leq r\leq+\infty, \quad c_1\in L^{p'}(\Omega), \end{cases} \tag{2.11}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, $\eta\in\mathbb{R}^N$.

Moreover $H:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $G:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying

$$\begin{cases} |H(x,s,\xi)|\leq b_0(x)|\xi|^\lambda+b_1(x), \\ \\ 0\leq\lambda\leq p-1, \quad b_0\in L^{N,1}(\Omega), \quad b_1\in L^1(\Omega), \end{cases} \tag{2.12}$$

$$G(x,s)s\geq 0, \tag{2.13}$$

$$\begin{cases} |G(x,s)|\leq d_1(x)|s|^t+d_2(x), \\ \\ d_1\in L^{z',1}(\Omega), \quad d_2\in L^1(\Omega), \end{cases} \tag{2.14}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$, where

$$0\leq t<\frac{N(p-1)}{N-p}, \quad z=\frac{N(p-1)}{N-p}\frac{1}{t} \text{ and } \frac{1}{z}+\frac{1}{z'}=1. \tag{2.15}$$

Finally μ is a measure in $M_b(\Omega)$ which is decomposed as

$$\mu=f-\mathrm{div}(g)+\mu_s^+-\mu_s^-, \tag{2.16}$$

according to Proposition 2.3, and

$$F\in\left(L^{p'}(\Omega)\right)^N. \tag{2.17}$$

Remark 2.3 Observe that, by (2.3) if the functions c_0 and b_0 belong to the Lebesgue spaces $L^t(\Omega)$ for some $t \geq \frac{N}{p-1}$ and $L^q(\Omega)$ for some $q > N$, then the conditions $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$, and $b_0 \in L^{N,1}(\Omega)$ (as requested in hypotheses (2.11) and (2.12)) are satisfied.

For $k > 0$, denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k, that is

$$T_k(s) = \begin{cases} s & |s| \leq k, \\ k \operatorname{sign}(s) & |s| > k. \end{cases}$$

Consider a measurable function $u : \Omega \to \bar{\mathbb{R}}$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists (see e.g. [BBGGPV], Lemma 2.1) an unique measurable function $v : \Omega \to \bar{\mathbb{R}}^N$, finite almost everywhere, such that

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \quad \text{almost everywhere in} \quad \Omega, \quad \forall k > 0. \tag{2.18}$$

We define the gradient ∇u of u as this function v, and denote $\nabla u = v$. Note that the previous definition does not coincide with the definition of the distributional gradient. However if $v \in (L^1_{loc}(\Omega))^N$, then $u \in W^{1,1}_{loc}(\Omega)$ and v is the distributional gradient of u. In contrast there are examples of functions $u \notin L^1_{loc}(\Omega)$ (and thus such that the gradient of u in the distributional sense is not defined) for which the gradient ∇u is defined in the previous sense (see Remarks 2.10 and 2.11, Lemma 2.12 and Example 2.16 in [DMOP]).

Definition 2.4 We say that a function $u : \Omega \to \bar{\mathbb{R}}$, measurable on Ω, almost everywhere finite, is a renormalized solution of (2.7) if it satisfies the following conditions

$$T_k(u) \in W_0^{1,p}(\Omega), \quad \forall k > 0; \tag{2.19}$$

$$|u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega); \tag{2.20}$$

$$|\nabla u|^{p-1} \quad \text{belongs to} \quad L^{N',\infty}(\Omega), \tag{2.21}$$

where ∇u is the gradient introduced in (2.18);

$$\lim_{n\to+\infty} \frac{1}{n} \int_{n<u<2n} a(x,u,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi\, d\mu_s^+, \tag{2.22}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{-2n<u<-n} a(x,u,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi\, d\mu_s^-, \tag{2.23}$$

for every $\varphi \in C_b^0(\Omega)$;

$$\lim_{n\to+\infty} \frac{1}{n} \int_{n<|u|<2n} |K(x,u)||\nabla u| = 0; \tag{2.24}$$

and finally

$$\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\cdot\nabla u\, h'(u)v+\int_\Omega a(x,u,\nabla u)\cdot\nabla v\, h(u)\\
&\quad+\int_\Omega K(x,u)\cdot\nabla u\, h'(u)v+\int_\Omega K(x,u)\cdot\nabla v\, h(u)\\
&\quad+\int_\Omega H(x,u,\nabla u)h(u)v+\int_\Omega G(x,u)h(u)v\\
&=\int_\Omega f h(u)v+\int_\Omega (g+F)\cdot\nabla u\, h'(u)v+\int_\Omega (g+F)\cdot\nabla v\, h(u),
\end{aligned}\tag{2.25}$$

for every $v\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, for every $h\in W^{1,\infty}(\mathbb{R})$ with a compact support in $\mathbb{R}$, which are such that $h(u)v\in W_0^{1,p}(\Omega)$.

Remark 2.5 Observe that every term in (2.25) is well-defined since $T_k(u)\in W_0^{1,p}(\Omega)$ for any $k>0$ and h has a compact support. In particular, since there exists $M>0$ (depending on h) such that $\text{supp}(h)\subset[-M,M]$,

$$\int_\Omega K(x,u)\cdot\nabla u\, h'(u)v=\int_\Omega K(x,T_M(u))\cdot\nabla T_M(u)\, h'(u)v. \tag{2.26}$$

Therefore such an integral is well defined thanks to the assumptions (2.11) and the facts that $T_M(u)\in W_0^{1,p}(\Omega)$ and h' are bounded.

Remark 2.6 Observe that every renormalized solution u of (2.7) is such that

$$|a(x,u,\nabla u)|\in L^{N',\infty}(\Omega),\qquad |K(x,u)|\in L^{N',r}(\Omega),\quad \frac{N}{p-1}\le r\le+\infty,$$

$$G(x,u)\in L^1(\Omega)\quad\text{and}\quad H(x,u,\nabla u)\in L^1(\Omega),$$

thanks to the conditions (2.20) and (2.21), and the growth conditions (2.9), (2.11), (2.12) and (2.14) on a, K, H and G respectively.

Observe also that, since $p<N$, then $L^{p'}(\Omega)\subset L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1}\le r\le+\infty$. Therefore the term $K(x,u)$ does not in general belong to $\left(L^{p'}(\Omega)\right)^N$ and the term $-\text{div}(K(x,u))$ is not in general an element of the dual space $W^{-1,p'}(\Omega)$.

Moreover u is a solution of (2.7) in the distributional sense, that is u satisfies

$$\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\cdot\nabla\phi+\int_\Omega K(x,u)\cdot\nabla\phi+\int_\Omega H(x,u,\nabla u)\phi+\int_\Omega G(x,u)\phi\\
&\qquad=\int_\Omega \phi\, d\mu+\int_\Omega F\cdot\nabla\phi,
\end{aligned}\tag{2.27}$$

for all $\phi \in C_0^\infty(\Omega)$.

This results follow from a standard technique, by taking $\phi \in C_0^\infty(\Omega)$ and h_n defined by

$$h_n(s) = \begin{cases} 0, & |s| > 2n \\ \frac{2n-|s|}{n}, & n < |s| \leq 2n \\ 1, & |s| \leq n, \end{cases} \tag{2.28}$$

in (2.25), and letting n tend to infinity.

3 Statement of existence result

The main result of the present paper is the following existence result

Theorem 3.1 *Under assumptions (2.8)-(2.17), there exists at least one renormalized solution u of (2.7) if one of the following conditions holds true*

1) $\gamma = p-1$, $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ *and* $\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)}$ *is small enough;*

2) $\gamma < p-1$ *and* $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$.

Remark 3.2 Observe that we assume that c_0 belongs $L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ in hypothesis 1), while c_0 belongs to $L^{\frac{N}{p-1},\infty}(\Omega)$ in hypothesis 2). This is due to the fact that we use the stability theorem Theorem 5.1 of [GM] in order to prove the existence result. Actually such a theorem holds true under the assumption that $\gamma = p-1$ and $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ or under the assumption that $\gamma < p-1$ and $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$.

Remark 3.3 The "limit case" where $\gamma = p-1$ and c_0 belongs to the Lorentz space $L^{\frac{N}{p-1},\infty}(\Omega)$ is not considered in Theorem 3.1. Actually we could prove an existence result under the assumptions that $\gamma = p-1$, $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$ with its norm in such a space small enough and the right-hand side μ a measure belonging to $M_0(\Omega)$ (and not more a general measure). This restrictions on the right-hand side in the case where c_0 belongs to $L^{\frac{N}{p-1},\infty}(\Omega)$ seems due to our method, which uses the stability result proved in [GM]. Indeed such a result can be proved for a class of problems of type (2.7) (with $G \equiv H \equiv 0$) under the assumptions (2.8)–(2.11) and (2.16) and (2.17) with $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$, $\mu \in M_0(\Omega)$, i.e. $\mu = f - \mathrm{div}(g)$ (see Remarks 4.2 and 4.6 in [GM]).

We will prove Theorem 3.1 by an approximation process. First the bounded Radon measure μ can be decomposed as

$$\mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-,$$

where $f \in L^1(\Omega)$, $g \in \left(L^{p'}(\Omega)\right)^N$ and μ_s^+ and μ_s^- (the positive and the negative parts of μ_s) are two nonnegative measures in $M_b(\Omega)$ which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity, according to Proposition 2.2.

As in [DMOP] (cf. [BMMP3]), we approximate the measure μ by a sequence μ_ε defined as

$$\mu_\varepsilon = f_\varepsilon - \operatorname{div}(g) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus,$$

where

$$\begin{cases} f_\varepsilon \quad \text{is a sequence of } L^{p'}(\Omega) \text{ functions} \\ \text{that converges to } f \text{ weakly in } L^1(\Omega), \end{cases} \tag{3.1}$$

$$\begin{cases} \lambda_\varepsilon^\oplus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \\ \text{that converges to } \mu_s^+ \text{ in the narrow topology of measures,} \end{cases} \tag{3.2}$$

and

$$\begin{cases} \lambda_\varepsilon^\ominus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \\ \text{that converges to } \mu_s^- \text{ in the narrow topology of measures.} \end{cases} \tag{3.3}$$

Observe that μ_ε belongs to $W^{-1,p'}(\Omega)$.

Let us denote by

$$K_\varepsilon(x,s) = K(x, T_{1/\varepsilon}(s)), \tag{3.4}$$

$$H_\varepsilon(x,s,\xi) = T_{1/\varepsilon}(H(x,s,\xi)), \tag{3.5}$$

$$G_\varepsilon(x,s) = T_{1/\varepsilon}(G(x,s)). \tag{3.6}$$

Therefore, by assumptions (2.11)-(2.14), we have

$$|K_\varepsilon(x,s)| \le |K(x,s)| \le c_0(x)|s|^\gamma + c_1(x), \tag{3.7}$$

$$|K_\varepsilon(x,s)| \le c_0(x)\frac{1}{\varepsilon^\gamma} + c_1(x), \tag{3.8}$$

$$|H_\varepsilon(x,s,\xi)| \le |H(x,s,\xi)| \le b_0(x)|\xi|^\lambda + b_1(x), \tag{3.9}$$

$$|H_\varepsilon(x,s,\xi)| \le \frac{1}{\varepsilon}, \tag{3.10}$$

$$G_\varepsilon(x,s)s \ge 0, \tag{3.11}$$

$$|G_\varepsilon(x,s)| \le |G(x,s)| \le d_1(x)|s|^r + d_2(x), \tag{3.12}$$

$$|G_\varepsilon(x,s)| \le \frac{1}{\varepsilon}. \tag{3.13}$$

Let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be a weak solution of the following problem

$$\begin{cases} -\mathrm{div}(a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon))+H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)+G_\varepsilon(x,u_\varepsilon)=\mu_\varepsilon-\mathrm{div}(F) & \text{in } \Omega \\ u_\varepsilon = 0. & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

i.e.

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v + \int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla v \\ \qquad \displaystyle + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v + \int_\Omega G_\varepsilon(x,u_\varepsilon)v \\ \displaystyle = \int_\Omega f_\varepsilon v + \int_\Omega (g+F)\cdot\nabla v + \int_\Omega \lambda_\varepsilon^\oplus v - \int_\Omega \lambda_\varepsilon^\ominus v, \\ \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.15}$$

The existence of a solution u_ε of (3.15) is a well-known result (see e.g. [L, DPo2]).

4 Proof of Theorem 3.1

The main difficulty in proving Theorem 3.1 is to obtain an a priori estimate of $|\nabla u_\varepsilon|^{p-1}$ in $L^{N',\infty}(\Omega)$. Let us explain our method in the case where $\gamma=\lambda=p-1$. By adapting a proof used in [G2], we decompose $|\nabla u_\varepsilon|^{p-1}$ in two terms

$$\begin{aligned} |\nabla u_\varepsilon|^{p-1} &= \chi_{\{|u_\varepsilon|>m_1\}}|\nabla u_\varepsilon|^{p-1} + \chi_{\{|u_\varepsilon|\le m_1\}}|\nabla u_\varepsilon|^{p-1} \\ &= |\nabla S_{m_1}(u_\varepsilon)|^{p-1} + |\nabla T_{m_1}(u_\varepsilon)|^{p-1}, \end{aligned}$$

where $S_{m_1}(u_\varepsilon)=u_\varepsilon - T_{m_1}(u_\varepsilon)$ is the "remainder" of the truncation $T_{m_1}(u_\varepsilon)$ and m_1 is a value suitably chosen. Then we firstly prove an a priori estimate for $|\nabla S_{m_1}(u_\varepsilon)|^{p-1}$, with a bound depending on m_1 and on the data; in this step we use a slighter generalization, proved in [BMMP3], of a result of [BBGGPV], which we state below

Lemma 4.1 *Assume that Ω is an open subset of $\mathbb{R}^N$ with finite measure and that $1<p<N$. Let u be a measurable function satisfying $T_k(u)\in W_0^{1,p}(\Omega)$, for every positive k, and such that*

$$\int_\Omega |\nabla T_k(u)|^p \le Mk + L, \quad \forall k>0, \tag{4.1}$$

where M and L are given constants. Then $|u|^{p-1}$ belongs to $L^{\frac{p^*}{p'},\infty}(\Omega)$, $|\nabla u|^{p-1}$ belongs to $L^{N',\infty}(\Omega)$ and

$$\||u|^{p-1}\|_{L^{\frac{p^*}{p'},\infty}(\Omega)} \le C(N,p)\left[M + |\Omega|^{\frac{1}{p^*}} L^{\frac{1}{p'}}\right], \tag{4.2}$$

$$\||\nabla u|^{p-1}\|_{L^{N',\infty}(\Omega)} \le C(N,p)\left[M + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}} L^{\frac{1}{p'}}\right], \tag{4.3}$$

where $C(N,p)$ is a constant depending only on N and p and where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$.

Secondly we give an a priori estimate of $|\nabla T_{m_1}(u_\varepsilon)|^{p-1}$ depending on m_1 and on the data. The third step is devoted to prove that m_1 is uniformly bounded by a constant which is independent on ε; this allows to obtain the a priori estimate of $|\nabla u_\varepsilon|^{p-1}$. In the last section we prove that the approximated terms $H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$ and $G_\varepsilon(x,u_\varepsilon)$ converge strongly in $L^1(\Omega)$; this allows us to reproduce the proof to the stability result (Theorem 5.1) proved in [GM], which is a slightly generalization of the stability result of [DMOP].

4.1 A priori estimates

The main step of the proof of Theorem 3.1 consists in proving an a priori estimate of $|\nabla u_\varepsilon|^{p-1}$ in $L^{N',\infty}(\Omega)$. Let us explicitly remark that such an a priori estimate holds true under more general assumptions on summability of c_0 of Theorem 3.1 (see Remark 4.3).

Theorem 4.2 *Under the assumptions of Theorem 3.1, every solution u_ε of (3.15) satisfies*

$$\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \le c, \tag{4.4}$$

$$\||u_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} \le c, \tag{4.5}$$

where c is a positive constant which depends only on p, $|\Omega|$, N, α, $\|b_0\|_{L^{N,1}(\Omega)}$, $\|b_1\|_{L^1(\Omega)}$, $\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $\|c_1\|_{L^{p'}(\Omega)}$, $\|g\|_{(L^{p'}(\Omega))^N}$, $\|F\|_{(L^{p'}(\Omega))^N}$, $\sup_\varepsilon \|f_\varepsilon\|_{L^1(\Omega)}$, $\sup_\varepsilon \left(\lambda_\varepsilon^\oplus(\Omega) + \lambda_\varepsilon^\ominus(\Omega)\right)$
and on the decreasing rearrangement $(b_0)^*$ of b_0.

Proof
We begin to prove Theorem 4.2 under assumption 1) of Theorem 3.1, i.e. when $\gamma = \lambda = p-1$ and $\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)}$ is small enough.

Observe that, since $r < +\infty$, $L^{\frac{N}{p-1},r}(\Omega) \subset L^{\frac{N}{p-1},\infty}(\Omega)$. Therefore c_0 belongs to $L^{\frac{N}{p-1},\infty}(\Omega)$ and, moreover, since we assume that $\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)}$ is small enough, we also have that $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small too. From now on we will use that

$$c_0 \in L^{\frac{N}{p-1},\infty}(\Omega) \quad \text{and} \quad \|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)} \text{ is small enough.} \tag{4.6}$$

As in [BMMP3], we define the following set Z_ε. As $|\Omega|$ is finite, the set of the constants c such that $|\{x \in \Omega,\ |u_\varepsilon(x)| = c\}| > 0$ is at most countable. Let Z_ε^c be the (countable) union of all those sets. Its complementary $Z_\varepsilon = \Omega - Z_\varepsilon^c$ is therefore the union of the sets such that $|\{x \in \Omega,\ |u_\varepsilon(x)| = c\}| = 0$. Since for every c

$$\nabla u_\varepsilon = 0 \qquad \text{a.e. on } \{x \in \Omega,\ |u_\varepsilon(x)| = c\},$$

and since Z_ε^c is at most a countable union, we obtain that

$$\nabla u_\varepsilon = 0 \quad \text{a.e. on } Z_\varepsilon^c. \tag{4.7}$$

First step. Using the techniques developed in [BMMP3], we give in this step an estimate on $S_{m_1}(u_\varepsilon)$ where m_1 is a positive real number depending on ε and on the data.

Define, for $m > 0$, the function $S_m : \mathbb{R} \to \mathbb{R}$ by

$$S_m(s) = s - T_m(s),$$

i.e.

$$S_m(s) = \begin{cases} 0 & |s| \le m, \\ (|s| - m)\operatorname{sign}(s) & |s| > m. \end{cases} \tag{4.8}$$

We use in (3.15) the test function $T_k(S_m(u_\varepsilon))$ and we obtain

$$\begin{aligned} &\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla T_k(S_m(u_\varepsilon)) + \int_\Omega K_\varepsilon(x, u_\varepsilon) \cdot \nabla T_k(S_m(u_\varepsilon)) \\ &\quad + \int_\Omega H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) T_k(S_m(u_\varepsilon)) + \int_\Omega G_\varepsilon(x, u_\varepsilon) T_k(S_m(u_\varepsilon)) \\ &= \int_\Omega f_\varepsilon T_k(S_m(u_\varepsilon)) + \int_\Omega (g + F) \cdot \nabla T_k(S_m(u_\varepsilon)) \\ &\quad + \int_\Omega \lambda_\varepsilon^\oplus T_k(S_m(u_\varepsilon)) - \int_\Omega \lambda_\varepsilon^\ominus T_k(S_m(u_\varepsilon)). \end{aligned} \tag{4.9}$$

Now we estimate the various terms in (4.9).

By the definition (4.8) of $S_m(s)$ and the ellipticity condition (2.8), we obtain

$$\begin{aligned} \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla T_k(S_m(u_\varepsilon)) &= \int_{\{m \le |u_\varepsilon| \le m+k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \\ &\ge \alpha \int_\Omega |\nabla T_k(S_m(u_\varepsilon))|^p. \end{aligned} \tag{4.10}$$

Let us now estimate $\left| \int_\Omega K_\varepsilon(x, u_\varepsilon) \cdot \nabla T_k(S_m(u_\varepsilon)) \right|$.

Let

$$\beta_p = \max\{1, 2^{p-1}\}.$$

By the definition (4.8) of $S_m(s)$, the growth condition (3.7) on K_ε, the generalized Sobolev inequality (2.6), the generalized Hølder inequality (2.4) and the Young inequality, we get

$$
\begin{aligned}
&\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla T_k(S_m(u_\varepsilon))\ \right| \\
&\quad\le \int_\Omega c_0|u_\varepsilon|^{p-1}|\nabla T_k(S_m(u_\varepsilon))| + \int_\Omega c_1|\nabla T_k(S_m(u_\varepsilon))| \\
&\quad\le \beta_p\int_\Omega c_0(|u_\varepsilon|-m)^{p-1}|\nabla T_k(S_m(u_\varepsilon))| + \beta_p m^{p-1}\int_\Omega c_0|\nabla T_k(S_m(u_\varepsilon))| \\
&\qquad + \int_\Omega c_1|\nabla T_k(S_m(u_\varepsilon))| \\
&\quad\le \beta_p\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|T_k(S_m(u_\varepsilon))\|^{p-1}_{L^{p^*,p}(\Omega)}\|\nabla T_k(S_m(u_\varepsilon))\|_{(L^p(\Omega))^N} \\
&\qquad +\beta_p m^{p-1}\|1\|_{L^{\frac{p^*}{p-1},p'}(\Omega)}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|\nabla T_k(S_m(u_\varepsilon))\|_{(L^p(\Omega))^N} \\
&\qquad +\|c_1\|_{L^{p'}(\Omega)}\|\nabla T_k(S_m(u_\varepsilon))\|_{(L^p(\Omega))^N} \\
&\quad\le \beta_p S_{N,p}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N} \\
&\qquad +\frac{\beta_p^{p'}}{p'}\|1\|^{p'}_{L^{\frac{p^*}{p-1},p'}(\Omega)}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}m^p + \frac{1}{p}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N} \\
&\qquad +\frac{2^{p'/p}}{p'\alpha^{p'/p}}\|c_1\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha}{2p}\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N} \\
&\quad= \left(\beta_p S_{N,p}+\frac{1}{p}\right)\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N} + \frac{\alpha}{2p}\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N} \\
&\qquad +\frac{\beta_p^{p'}}{p'}\|1\|^{p'}_{L^{\frac{p^*}{p-1},p'}(\Omega)}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}m^p + \frac{2^{p'/p}}{p'\alpha^{p'/p}}\|c_1\|^{p'}_{L^{p'}(\Omega)}.
\end{aligned}
\tag{4.11}
$$

Let us now estimate $\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)T_k(S_m(u_\varepsilon))\right|$.

By the definition (4.8) of S_m, the growth assumption (3.9) on H_ε and the generalized Hølder inequality (2.4), we have

$$\begin{aligned}\Big|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)T_k(S_m(u_\varepsilon))\Big|&\\ &\le k\int_{\{|u_\varepsilon|>m\}}|H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\\ &\le k\left[\int_{\{|u_\varepsilon|>m\}}b_0|\nabla u_\varepsilon|^{p-1}+\int_\Omega b_1\right]\\ &= k\left[\int_{Z_\varepsilon\cap\{|u_\varepsilon|>m\}}b_0|\nabla S_m(u_\varepsilon)|^{p-1}+\int_\Omega b_1\right]\\ &\le k\left[\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{u_\varepsilon>m\})}\||\nabla S_m(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)}+\|b_1\|_{L^1(\Omega)}\right].\end{aligned} \tag{4.12}$$

Moreover, by the "sign condition" (3.11) on G_ε, we get

$$\int_\Omega G_\varepsilon(x,u_\varepsilon)T_k(S_m(u_\varepsilon))\ge 0. \tag{4.13}$$

Finally

$$\int_\Omega f_\varepsilon T_k(S_m(u_\varepsilon))\le k\|f_\varepsilon\|_{L^1(\Omega)}, \tag{4.14}$$

$$\int_\Omega(g+F)\cdot\nabla T_k(S_m(u_\varepsilon))\le\frac{\alpha}{2p}\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N}+\frac{2^{p'/p}}{p'\alpha^{p'/p}}\|g+F\|^{p'}_{(L^{p'}(\Omega))^N}, \tag{4.15}$$

$$\Big|\int_\Omega\lambda^\oplus_\varepsilon T_k(S_m(u_\varepsilon))\Big|\le k\lambda^\oplus_\varepsilon(\Omega), \tag{4.16}$$

$$\Big|\int_\Omega\lambda^\ominus_\varepsilon T_k(S_m(u_\varepsilon))\Big|\le k\lambda^\ominus_\varepsilon(\Omega). \tag{4.17}$$

Denote by

$$C_1=\frac{\alpha}{p'}-(\beta_pS_{N,p}+\frac{1}{p})\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}. \tag{4.18}$$

Observe that, since $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough, from now on we can assume

$$\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}<\frac{p\alpha}{p'(\beta_p pS_{N,p}+1)}, \tag{4.19}$$

so that C_1 is a positive constant.

Combining (4.9)-(4.17), we get

$$\|\nabla T_k(S_m(u_\varepsilon))\|^p_{(L^p(\Omega))^N} \le Mk + L, \qquad \forall k > 0, \tag{4.20}$$

where M and L are defined by

$$\begin{cases} M = \frac{1}{C_1}\left(\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\||\nabla S_m(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)} + M_0\right), \\ M_0 = \|b_1\|_{L^1(\Omega)} + \sup_\varepsilon \|f_\varepsilon\|_{L^1(\Omega)} + \sup_\varepsilon \left[\lambda^\oplus_\varepsilon(\Omega) + \lambda^\ominus_\varepsilon(\Omega)\right], \end{cases} \tag{4.21}$$

$$\begin{cases} L = L_1 m^p + L_0, \\ L_1 = \dfrac{1}{C_1}\dfrac{\beta_p^{p'}}{p'}\|1\|^{p'}_{L^{\frac{p^*}{p-1},p'}(\Omega)}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}, \\ L_0 = \dfrac{1}{C_1}\dfrac{2^{p'/p}}{p'\alpha^{p'/p}}\left(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}\right). \end{cases} \tag{4.22}$$

By Lemma 4.1, we get

$$\begin{aligned} &\||\nabla S_m(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)} \\ &\quad \le C(N,p)\left[M + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}}L^{\frac{1}{p'}}\right] \\ &\quad \le C'(N,p)\left[\frac{1}{C_1}\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\||\nabla S_m(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)}\right. \\ &\qquad \left. + \frac{M_0}{C_1} + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}}L_1^{\frac{1}{p'}}m^{p-1} + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}}L_0^{\frac{1}{p'}}\right]. \end{aligned} \tag{4.23}$$

Denote by $\left(b_{0_{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}}}\right)^*$ and $(b_0)^*$ the decreasing rearrangements of the restriction $b_{0_{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}}}$ and of b_0, respectively.
By the definition (2.1) of norm of Lorentz spaces and the definition of the decreasing rearrangement, it is easy to verify that the following inequality holds true

$$\begin{aligned} \|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{|u_\varepsilon|>m\})} &= \int_0^{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}|}\left(b_{0_{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}}}\right)^*(t)\,t^{\frac{1}{N}}\frac{dt}{t} \\ &\le \int_0^{|Z_\varepsilon\cap\{|u_\varepsilon|>m\}|}(b_0)^*(t)\,t^{1/N}\frac{dt}{t}. \end{aligned} \tag{4.24}$$

In the case where

$$\frac{C(N,p)}{C_1}\|b_0\|_{L^{N,1}(Z_\varepsilon)} = \frac{C(N,p)}{C_1}\int_0^{|Z_\varepsilon|}(b_0)^*(t)\,t^{1/N}\frac{dt}{t} \le \frac{1}{2}, \tag{4.25}$$

we choose $m = m_1 = 0$ and the proof is completed . Let us assume that (4.25) does not hold. Since the function $m \to |Z_\varepsilon \cap \{|u_\varepsilon| > m\}|$ is continuous (indeed the constants c such that the sets $\{|u_\varepsilon(x)| = c\}$ have a strictly positive measure have been eliminated by considering Z_ε), decreasing and tends to 0 when m tends to ∞, we can choose $m = m_1 > 0$ such that

$$\frac{C(N,p)}{C_1}\int_0^{|Z_\varepsilon\cap\{|u_\varepsilon|>m_1\}|}(b_0)^*(t)\,t^{1/N}\frac{dt}{t} = \frac{1}{2}.$$

Moreover, we define δ by

$$\frac{C(N,p)}{C_1}\int_0^{\delta}(b_0)^*(t)\,t^{1/N}\frac{dt}{t} = \frac{1}{2}. \tag{4.26}$$

Then we have

$$|Z_\varepsilon \cap \{|u_\varepsilon| > m_1\}| = \delta. \tag{4.27}$$

Observe that δ does not depend on ε.

Choosing $m = m_1$, we obtain from (4.23)

$$\| |\nabla S_{m_1}(u_\varepsilon)|^{p-1} \|_{L^{N',\infty}(\Omega)} \le 2C(N,p)\left[\frac{M_0}{C_1} + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}} L_1^{\frac{1}{p'}} m_1^{p-1} + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}} L_0^{\frac{1}{p'}}\right], \tag{4.28}$$

where M_0, L_0 and L_1 are defined by (4.21) and (4.22).

Second step. We now give an estimate on $T_{m_1}(u_\varepsilon)$.

Using in (3.15) the test function $T_{m_1}(u_\varepsilon)$, we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla T_{m_1}(u_\varepsilon) + \int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla T_{m_1}(u_\varepsilon) \\
&\quad + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,T_{m_1}(u_\varepsilon) + \int_\Omega G_\varepsilon(x,u_\varepsilon)\,T_{m_1}(u_\varepsilon) \\
&= \int_\Omega f_\varepsilon T_{m_1}(u_\varepsilon) + \int_\Omega (g+F)\cdot\nabla T_{m_1}(u_\varepsilon) \\
&\quad + \int_\Omega \lambda_\varepsilon^{\oplus} T_{m_1}(u_\varepsilon) - \int_\Omega \lambda_\varepsilon^{\ominus} T_{m_1}(u_\varepsilon).
\end{aligned} \tag{4.29}$$

Now we evaluate the various terms in (4.29).

By the ellipticity condition (2.8), we obtain

$$\begin{aligned}
\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla T_{m_1}(u_\varepsilon) &= \int_{\{|u_\varepsilon|\le m_1\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon \\
&\ge \alpha\int_\Omega |\nabla T_{m_1}(u_\varepsilon)|^p.
\end{aligned} \tag{4.30}$$

Let us now estimate $\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla T_{m_1}(u_\varepsilon)\right|$.

By the growth condition (3.7) on K_ε, the generalized Sobolev inequality (2.6), the generalized Hølder inequality (2.4) and the Young inequality, we get

$$
\begin{aligned}
&\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla T_{m_1}(u_\varepsilon)\right| \\
&\quad\le \int_\Omega c_0|u_\varepsilon|^{p-1}|\nabla T_{m_1}(u_\varepsilon)| + \int_\Omega c_1|\nabla T_{m_1}(u_\varepsilon)| \\
&\quad\le \|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|T_{m_1}(u_\varepsilon)\|^{p-1}_{L^{p^*,p}(\Omega)}\|\nabla T_{m_1}(u_\varepsilon)\|_{(L^p(\Omega))^N} \\
&\qquad+\|c_1\|_{L^{p'}(\Omega)}\|\nabla T_{m_1}(u_\varepsilon)\|_{(L^p(\Omega))^N} \\
&\quad\le S_{N,p}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\|\nabla T_{m_1}(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\qquad+\frac{4^{p'/p}}{p'\alpha^{p'/p}}\|c_1\|^{p'}_{L^{p'}(\Omega)}+\frac{\alpha}{4p}\|\nabla T_{m_1}(u_\varepsilon)\|^p_{(L^p(\Omega))^N}.
\end{aligned}
\tag{4.31}
$$

Let us now estimate $\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)T_{m_1}(u_\varepsilon)\right|$.

By the growth assumption (3.9) on H_ε and the generalized Hølder inequality (2.4), we have

$$
\begin{aligned}
&\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)T_{m_1}(u_\varepsilon)\right| \le \\
&\quad\le m_1\int_\Omega |H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)| \\
&\quad\le m_1\left[\int_{\{|u_\varepsilon|\le m_1\}} b_0|\nabla u_\varepsilon|^{p-1}+\int_{\{|u_\varepsilon|>m_1\}} b_0|\nabla u_\varepsilon|^{p-1}+\int_\Omega b_1\right] \\
&\quad\le \frac{2^{p/p'}m_1^p}{p\alpha^{p/p'}}\|b_0\|^p_{L^p(\Omega)}+\frac{\alpha}{2p'}\|\nabla T_{m_1}(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\qquad+m_1\left[\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{u_\varepsilon>m_1\})}\||\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)}+\|b_1\|_{L^1(\Omega)}\right].
\end{aligned}
\tag{4.32}
$$

Moreover, by the “sign condition” (3.11) on G_ε, we get

$$\int_\Omega G_\varepsilon(x,u_\varepsilon)T_{m_1}(u_\varepsilon)\ge 0. \tag{4.33}$$

Finally we have

$$\int_\Omega f_\varepsilon T_{m_1}(u_\varepsilon)\le m_1\|f_\varepsilon\|_{L^1(\Omega)}, \tag{4.34}$$

$$\int_\Omega (g+F)\cdot\nabla T_{m_1}(u_\varepsilon)\le\frac{\alpha}{4p}\|\nabla T_{m_1}(u_\varepsilon)\|^p_{(L^p(\Omega))^N}+\frac{4^{p'/p}}{p'\alpha^{p'/p}}\|g+F\|^{p'}_{\left(L^{p'}(\Omega)\right)^N}, \tag{4.35}$$

$$\left|\int_\Omega \lambda_\varepsilon^\oplus T_{m_1}(u_\varepsilon)\right| \le m_1 \lambda_\varepsilon^\oplus(\Omega), \tag{4.36}$$

$$\left|\int_\Omega \lambda_\varepsilon^\ominus T_{m_1}(u_\varepsilon)\right| \le m_1 \lambda_\varepsilon^\ominus(\Omega). \tag{4.37}$$

Denote by

$$C_2 = \frac{\alpha}{2} - S_{N,p}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}. \tag{4.38}$$

Since $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough, from now on we can suppose

$$\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)} < \frac{\alpha}{2S_{N,p}}, \tag{4.39}$$

so that C_2 is a positive constant (recall that the norm of c_0 also satisfies (4.19)).

Combining (4.29)-(4.37), we get

$$\begin{aligned}
&\|\nabla T_{m_1}(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\le \frac{1}{C_2}\left\{ \frac{2^{p/p'} m_1^p}{p\alpha^{p/p'}}\|b_0\|^p_{L^p(\Omega)} \right. \\
&+ m_1\left[\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{u_\varepsilon>m_1\})}\||\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)} + M_0\right] \\
&\left. + \frac{4^{p'/p}}{p'\alpha^{p'/p}}\left(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}\right)\right\},
\end{aligned} \tag{4.40}$$

where M_0 is defined by (4.21).

Third step. In this step we prove that m_1 is uniformly bounded with respect to ε. It is performed through a technical "log–type" estimate on u_ε (cfr. [B], [G2]).

To this end, let us define for $h > 0$ the function $\phi_h : \mathbb{R} \to \mathbb{R}$ by

$$\phi_h(s) = \left\{\frac{1}{\left[(h+1)m_1 - |T_{m_1}(s)|\right]^{p-1}} - \frac{1}{[(h+1)m_1]^{p-1}}\right\}\operatorname{sign}(s),$$

i.e.

$$\phi_h(s) = \begin{cases} \left[\dfrac{1}{((h+1)m_1 - |s|)^{p-1}} - \dfrac{1}{[(h+1)m_1]^{p-1}}\right]\operatorname{sign}(s), & |s| \le m_1, \\[2ex] \left[\dfrac{1}{(hm_1)^{p-1}} - \dfrac{1}{((h+1)m_1)^{p-1}}\right]\operatorname{sign}(s), & |s| > m_1. \end{cases} \tag{4.41}$$

Observe that the following property of $\phi_h(s)$ holds true

$$|\phi_h(s)| \le \frac{1}{(hm_1)^{p-1}}, \qquad \forall s \in \mathbb{R}. \tag{4.42}$$

Since $\phi_h(s)$ is a Lipschitz continuous function with $\phi_h(0)=0$ and since $u_\varepsilon \in W_0^{1,p}(\Omega)$, the function $\phi_h(u_\varepsilon)$ belongs to $W_0^{1,p}(\Omega)$. This allows us to use $\phi_h(u_\varepsilon)$ as a test function in (3.15). Then we get

$$\begin{aligned}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\phi_h'(u_\varepsilon)+\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\phi_h'(u_\varepsilon)&\\ +\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\phi_h(u_\varepsilon)+\int_\Omega G_\varepsilon(x,u_\varepsilon)\phi_h(u_\varepsilon)&\\ =\int_\Omega f_\varepsilon\phi_h(u_\varepsilon)+\int_\Omega(g+F)\cdot\nabla u_\varepsilon\phi_h'(u_\varepsilon)+\int_\Omega\lambda_\varepsilon^\oplus\phi_h(u_\varepsilon)-\int_\Omega\lambda_\varepsilon^\ominus\phi_h(u_\varepsilon).&\end{aligned}\tag{4.43}$$

Now we estimate the various integrals in (4.43).

By the definition (4.41) of $\phi_h(s)$ and the ellipticity condition (2.8), we obtain

$$\begin{aligned}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\phi_h'(u_\varepsilon)\geq&\int_{\{|u_\varepsilon|\leq m_1\}}a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\phi_h'(u_\varepsilon)\\ &\geq(p-1)\alpha\int_\Omega\frac{|\nabla T_{m_1}(u_\varepsilon)|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}.\end{aligned}\tag{4.44}$$

Let us now estimate $\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\phi_h'(u_\varepsilon)\nabla u_\varepsilon\right|$.

Since, $m_1-|T_{m_1}(s)|\geq 0$ for any $s\in\mathbb{R}$, the growth condition (3.7) on K_ε and the Young inequality yield

$$\begin{aligned}&\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\phi_h'(u_\varepsilon)\right|\\ &\leq\int_\Omega c_0|u_\varepsilon|^{p-1}|\nabla u_\varepsilon||\phi_h'(u_\varepsilon)|+\int_\Omega c_1|\nabla u_\varepsilon||\phi_h'(u_\varepsilon)|\\ &=(p-1)\int_{|u_\varepsilon|\leq m_1}\frac{c_0|u_\varepsilon|^{p-1}|\nabla u_\varepsilon|}{[(h+1)m_1-|u_\varepsilon|]^p}+(p-1)\int_{|u_\varepsilon|\leq m_1}\frac{c_1|\nabla u_\varepsilon|}{[(h+1)m_1-|u_\varepsilon|]^p}\\ &\leq\frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}}\int_\Omega\frac{c_0^{p'}m_1^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}+\frac{(p-1)\alpha}{3p}\int_\Omega\frac{|\nabla u_\varepsilon|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}\\ &\quad+\frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}}\int_\Omega\frac{c_1^{p'}}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}+\frac{(p-1)\alpha}{3p}\int_\Omega\frac{|\nabla u_\varepsilon|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}\\ &\leq\frac{2(p-1)\alpha}{3p}\int_\Omega\frac{|\nabla u_\varepsilon|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}\\ &\quad+\frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}}\left(\frac{1}{h^p}\|c_0\|_{L^{p'}(\Omega)}^{p'}+\frac{1}{(hm_1)^p}\|c_1\|_{L^{p'}(\Omega)}^{p'}\right).\end{aligned}$$

Moreover, since $p < N$, we have $L^{\frac{N}{p-1},\infty}(\Omega) \subset L^{p'}(\Omega)$ and by inequality (2.5) it follows that

$$\|c_0\|_{L^{p'}(\Omega)} \le \|1\|_{L^{\frac{pN}{(p-1)(N-p)},p'}(\Omega)} \|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$$

i.e.

$$\|c_0\|_{L^{p'}(\Omega)} \le \frac{N}{N-p} |\Omega|^{\frac{N-p}{N}} \|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}.$$

Therefore, we obtain

$$\begin{aligned} &\Big| \int_\Omega K_\varepsilon(x,u_\varepsilon) \cdot \nabla u_\varepsilon \phi'_h(u_\varepsilon) \Big| \\ &\le \frac{2(p-1)\alpha}{3p} \int_\Omega \frac{|\nabla u_\varepsilon|^p}{\left[(h+1)m_1 - |T_{m_1}(u_\varepsilon)|\right]^p} \\ &\quad + \frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}} \left[\frac{1}{h^p} \left(\frac{N}{N-p}\right)^{p'} |\Omega|^{\frac{(N-p)p'}{N}} \|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)} + \frac{1}{(hm_1)^p} \|c_1\|^{p'}_{L^{p'}(\Omega)} \right]. \end{aligned} \tag{4.45}$$

Let us now estimate $\left| \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\phi_h(u_\varepsilon) \right|$.

By the definition (4.41) of ϕ_h, the growth assumption (3.9) on H_ε, the property (4.42) and the generalized Hølder inequality (2.4), we have

$$\begin{aligned} &\Big| \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\phi_h(u_\varepsilon) \Big| \\ &\quad \le \int_\Omega b_0 |\nabla u_\varepsilon|^{p-1} |\phi_h(u_\varepsilon)| + \int_\Omega b_1 |\phi_h(u_\varepsilon)| \\ &\quad \le \int_{Z_\varepsilon \cap \{|u_\varepsilon| \le m_1\}} \frac{b_0 |\nabla u_\varepsilon|^{p-1}}{[(h+1)m_1 - |u_\varepsilon|]^{p-1}} \\ &\qquad + \frac{1}{(hm_1)^{p-1}} \int_{Z_\varepsilon \cap \{|u_\varepsilon| > m_1\}} b_0 |\nabla u_\varepsilon|^{p-1} + \frac{1}{(hm_1)^{p-1}} \|b_1\|_{L^1(\Omega)} \\ &\quad \le \frac{2^{p/p'}}{[p(p-1)\alpha]^{p/p'}} \|b_0\|^p_{L^p(\Omega)} + \frac{(p-1)\alpha}{2p'} \int_{|u_\varepsilon| \le m_1} \frac{|\nabla u_\varepsilon|^p}{\left[(h+1)m_1 - |T_{m_1}(u_\varepsilon)|\right]^p} \\ &\qquad + \frac{1}{(hm_1)^{p-1}} \left[\|b_0\|_{L^{N,1}(Z_\varepsilon \cap \{u_\varepsilon > m\})} \| |\nabla S_{m_1}(u_\varepsilon)|^{p-1} \|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)} \right]. \end{aligned} \tag{4.46}$$

Since $p < N$, we have $L^{N,1}(\Omega) \subset L^p(\Omega)$ and therefore the coefficient b_0 belongs to $L^p(\Omega)$.

Moreover, by the "sign condition" (3.11) of G_ε, we get

$$\int_\Omega G_\varepsilon(x,u_\varepsilon)\phi_h(u_\varepsilon) \ge 0. \tag{4.47}$$

Finally, since for any $s \in \mathbb{R}$ we have $(h+1)m_1 - |T_{m_1}(s)| \geq hm_1$, we get

$$\begin{aligned}\int_\Omega (g+F)\cdot\nabla u_\varepsilon \phi_h'(u_\varepsilon) &= (p-1)\int_{|u_\varepsilon|\leq m_1} \frac{(g+F)\cdot\nabla u_\varepsilon}{[(h+1)m_1-|u_\varepsilon|]^p} \\ &\leq \frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}(hm_1)^p}\|g+F\|^{p'}_{(L^{p'}(\Omega))^N} + \frac{(p-1)\alpha}{3p}\int_\Omega \frac{|\nabla u_\varepsilon|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}\end{aligned} \tag{4.48}$$

and, the property (4.42) of ϕ_h gives that

$$\int_\Omega f_\varepsilon \phi_h(u_\varepsilon) \leq \frac{1}{(hm_1)^{p-1}}\|f_\varepsilon\|_{L^1(\Omega)}, \tag{4.49}$$

$$\left|\int_\Omega \lambda_\varepsilon^\oplus \phi_h(u_\varepsilon)\right| \leq \frac{1}{(hm_1)^{p-1}}\lambda_\varepsilon^\oplus(\Omega), \tag{4.50}$$

$$\left|\int_\Omega \lambda_\varepsilon^\ominus \phi_h(u_\varepsilon)\right| \leq \frac{1}{(hm_1)^{p-1}}\lambda_\varepsilon^\ominus(\Omega). \tag{4.51}$$

Gathering (4.43)-(4.51) leads to

$$\begin{aligned}&\int_\Omega \frac{|\nabla T_{m_1}(u_\varepsilon)|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p} \\ &\leq C(p,N,|\Omega|,\alpha)\Bigg\{\|b_0\|^p_{L^p(\Omega)} + \frac{1}{(hm_1)^{p-1}}\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{u_\varepsilon>m_1\})}\||\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)} \\ &\quad + \frac{1}{(hm_1)^{p-1}}M_0 + \frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}) + \frac{1}{h^p}\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)}\Bigg\},\end{aligned}$$

where M_0 is defined by (4.21) and where

$$\begin{aligned}C(p,N,|\Omega|,\alpha) = \frac{2p'}{(p-1)\alpha}\max\Bigg\{&\frac{2^{p/p'}}{[p(p-1)\alpha]^{p/p'}}, 1, \\ &\frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}}, \frac{3^{p'/p}(p-1)}{p'\alpha^{p'/p}}\left(\frac{N}{N-p}\right)^{p'}|\Omega|^{\frac{(N-p)p'}{N}}\Bigg\}.\end{aligned}$$

On the one hand using the estimate (4.28) of $|\nabla S_{m_1}(u_\varepsilon)|^{p-1}$ in the first step together with the Young inequality and the definition (4.22) of L_1 yields that

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla T_{m_1}(u_\varepsilon)|^p}{\left[(h+1)m_1-|T_{m_1}(u_\varepsilon)|\right]^p}\\
&\le C'(p,N,|\Omega|,\alpha)\left\{\|b_0\|^p_{L^p(\Omega)}+\frac{1}{(hm_1)^{p-1}}\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{u_\varepsilon>m_1\})}\left[M_0+L_1^{\frac{1}{p'}}m_1^{p-1}+L_0^{\frac{1}{p'}}\right]\right.\\
&\quad\left.+\frac{1}{(hm_1)^{p-1}}M_0+\frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)}+\|g+F\|^{p'}_{(L^{p'}(\Omega))^N})+\frac{1}{h^p}\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)}\right\}\\
&\le C'(p,N,|\Omega|,\alpha)\left\{\|b_0\|^p_{L^p(\Omega)}+\frac{1}{h^{p-1}}\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{u_\varepsilon>m_1\})}L_1^{\frac{1}{p'}}\right.\\
&\quad+\frac{1}{(hm_1)^{p-1}}\left[\|b_0\|_{L^{N,1}(\Omega)}(M_0+L_0^{\frac{1}{p'}})+M_0\right]\\
&\quad\left.+\frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)}+\|g+F\|^{p'}_{(L^{p'}(\Omega))^N})+\frac{1}{h^p}\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)}\right\}\\
&\le C'(p,N,|\Omega|,\alpha)\left\{\|b_0\|^p_{L^p(\Omega)}+\frac{1}{p}\|b_0\|^p_{L^{N,1}(\Omega)}+\frac{1}{p'h^p}L_1\right.\\
&\quad+\frac{1}{(hm_1)^{p-1}}\left[\|b_0\|_{L^{N,1}(\Omega)}(M_0+L_0^{\frac{1}{p'}})+M_0\right]\\
&\quad\left.+\frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)}+\|g+F\|^{p'}_{(L^{p'}(\Omega))^N})+\frac{1}{h^p}\|c_0\|^{p'}_{L^{\frac{N}{p-1}\infty}(\Omega)}\right\},\\
&\le C'(p,N,|\Omega|,\alpha)\left\{\|b_0\|^p_{L^p(\Omega)}+\frac{1}{p}\|b_0\|^p_{L^{N,1}(\Omega)}+\frac{1}{(hm_1)^{p-1}}\left[\|b_0\|_{L^{N,1}(\Omega)}(M_0+L_0^{\frac{1}{p'}})+M_0\right]\right.\\
&\quad\left.+\frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)}+\|g+F\|^{p'}_{(L^{p'}(\Omega))^N})+\frac{1}{h^p}\left(\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)}+\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\right)\right\},
\end{aligned}
\tag{4.52}
$$

where L_0 and L_1 are defined by (4.22) and where

$$
C'(p,N,|\Omega|,\alpha)=C(p,N,|\Omega|,\alpha)\max\left\{1,2C(N,p)\left(\frac{\beta_p^{p'}}{C_1p'^2}\|1\|_{L^{\frac{p^*}{p-1},p'}}+1\right)\right\}.
$$

On the other hand, due to (4.27), we have

$$
\text{meas}(Z_\varepsilon\cap\{u_\varepsilon>m_1\})=\delta\le\text{meas}(\{u_\varepsilon>m_1\}),
$$

and by the Poincaré inequality, we get

$$\begin{aligned}
&\int_\Omega \frac{|\nabla T_{m_1}(u_\varepsilon)|^p}{\left[(h+1)m_1 - |T_{m_1}(u_\varepsilon)|\right]^p} = \int_\Omega \left|\nabla \log\left[(h+1)(m_1+1) - |T_{m_1}(u_\varepsilon)|\right]\right|^p \\
&= \int_\Omega \left|\nabla \log\left[1 - \frac{|T_{m_1}(u_\varepsilon)|}{(h+1)m_1}\right]\right|^p \\
&\geq c(|\Omega|, p)\int_\Omega \left|\log\left[1 - \frac{|T_{m_1}(u_\varepsilon)|}{(h+1)m_1}\right]\right|^p \\
&\geq c(|\Omega|, p)\int_{|u_\varepsilon|>m_1} \left|\log\left[1 - \frac{|T_{m_1}(u_\varepsilon)|}{(h+1)m_1}\right]\right|^p \\
&\geq c(|\Omega|, p)\left|\log\left[1 - \frac{m_1}{(h+1)m_1}\right]\right|^p \delta \\
&= c(|\Omega|, p)\left[\log\left(1 + \frac{1}{h}\right)\right]^p \delta.
\end{aligned} \tag{4.53}$$

Combining (4.52) and (4.53), we obtain

$$\begin{aligned}
\left[\log\left(1+\frac{1}{h}\right)\right]^p \leq &\frac{C''(p,N,|\Omega|,\alpha)}{\delta}\left\{\|b_0\|^p_{L^p(\Omega)} + \frac{1}{p}\|b_0\|^p_{L^{N,1}(\Omega)}\right. \\
&+ \frac{1}{(hm_1)^{p-1}}\left[\|b_0\|_{L^{N,1}(\Omega)}(M_0 + L_0^{\frac{1}{p'}}) + M_0\right] \\
&+ \frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}) \\
&\left. + \frac{1}{h^p}\left(\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)} + \|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\right)\right\},
\end{aligned} \tag{4.54}$$

We are now in a position to prove that m_1 is uniformly bounded with respect to ε by a suitable choice of h in (4.54) and if $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough. We first fix $h = h_1$ such that

$$\frac{C''(p,N,|\Omega|,\alpha)}{\delta}\left(\|b_0\|^p_{L^p(\Omega)} + \frac{1}{p}\|b_0\|^p_{L^{N,1}(\Omega)}\right) = \frac{1}{2}\left[\log\left(1+\frac{1}{h_1}\right)\right]^p.$$

Observe that h_1 is independent on ε. Therefore we get from (4.54)

$$\begin{aligned}
&\left[\log\left(1+\frac{1}{h}\right)\right]^p \\
&\leq \frac{2C''(p,N,|\Omega|,\alpha)}{\delta}\left\{\frac{1}{(hm_1)^{p-1}}\left[\|b_0\|_{L^{N,1}(\Omega)}(M_0 + L_0^{\frac{1}{p'}}) + M_0\right]\right. \\
&\quad + \frac{1}{(hm_1)^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}) \\
&\quad \left. + \frac{1}{h^p}\left(\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)} + \|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\right)\right\}.
\end{aligned} \tag{4.55}$$

Denote

$$a_1 = \left[\log\left(1+\frac{1}{h_1}\right)\right]^p - \frac{2C''(p,N,|\Omega|,\alpha)}{\delta h_1^p}\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)},$$

$$a_2 = \frac{2C''(p,N,|\Omega|,\alpha)}{\delta h_1^{p-1}}\left[\|b_0\|_{L^{N,1}(\Omega)}(M_0+L_0^{\frac{1}{p'}})+M_0\right],$$

$$a_3 = \frac{2C''(p,N,|\Omega|,\alpha)}{\delta h_1^p}(\|c_1\|^{p'}_{L^{p'}(\Omega)}+\|g+F\|^{p'}_{(L^{p'}(\Omega))^N}).$$

Since $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough, we can assume

$$\frac{2C''(p,N,|\Omega|,\alpha)}{\delta h_1^p}\left(\|c_0\|^{p'}_{L^{\frac{N}{p-1},\infty}(\Omega)}+\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}\right) < \left[\log\left(1+\frac{1}{h_1}\right)\right]^p,$$

so that a_1 is a positive constant (recall that the norm of c_0 also satisfies (4.19) and (4.39)).

Observe also that a_1, a_2 and a_3 are constants independent on ε.

Therefore by (4.55) we get

$$a_1 < \frac{a_2}{m_1^{p-1}} + \frac{a_3}{m_1^p},$$

which allows to conclude that

$$m_1 \le c \tag{4.56}$$

where c is a constant which does not depends on ε [1].

By the estimate (4.28) of $|\nabla S_{m_1}(u_\varepsilon)|^{p-1}$ in the first step, we deduce that

$$\||\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|^p_{L^p(\Omega)} \le c, \tag{4.57}$$

and therefore by the estimate (4.40) of $\nabla T_{m_1}(u_\varepsilon)$ in the second step, we get also

$$\|\nabla T_{m_1}(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \le c. \tag{4.58}$$

Moreover, writing

$$\begin{aligned}|\nabla u_\varepsilon|^{p-1} &= |\nabla u_\varepsilon|^{p-1}\chi_{\{|u_\varepsilon|\le m_1\}} + |\nabla u_\varepsilon|^{p-1}\chi_{\{|u_\varepsilon|>m_1\}}\\ &= |\nabla T_{m_1}(u_\varepsilon)|^{p-1} + |\nabla S_{m_1}(u_\varepsilon)|^{p-1},\end{aligned}$$

and using (4.57) and (4.58) lead to

$$\begin{aligned}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} &\le \||\nabla T_{m_1}(u_\varepsilon)|^{p-1}\|_{(L^{N',\infty}(\Omega))^N} + \||\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)}\\ &\le c\|\nabla T_{m_1}(u_\varepsilon)\|_{(L^p(\Omega))^N} + \||\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)} \le c,\end{aligned}$$

[1] From now on c will be denote a constant which depends only on the data of the problem, but which does not depends on ε and which can vary from line to line.

that is (4.4).

We now turn to inequality (4.5). We observe that (4.20) holds true also with $m = m_1$. Therefore by Lemma 4.1, and the estimates (4.56) and (4.57), we get

$$\begin{aligned}
&\| |S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} \\
&\le C(N,p)\left[M + |\Omega|^{\frac{1}{p^*}} L^{\frac{1}{p'}}\right] \\
&= C(N,p)\left[\frac{1}{C_1}\|b_0\|_{L^{N,1}(Z_\varepsilon\cap\{|u_\varepsilon|>m\})}\| |\nabla S_{m_1}(u_\varepsilon)|^{p-1}\|_{L^{N',\infty}(\Omega)}\right. \\
&\quad \left. + \frac{1}{C_1}M_0 + |\Omega|^{\frac{1}{p^*}} L_1^{\frac{1}{p'}} m_1^{p-1} + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}} L_0^{\frac{1}{p'}}\right] \le c.
\end{aligned} \tag{4.59}$$

where M, L, M_0, L_0 and L_1 are defined by (4.21) and (4.22) respectively.

Moreover, we have

$$\begin{aligned}
|u_\varepsilon|^{p-1} &= |u_\varepsilon|^{p-1}\chi_{\{|u_\varepsilon|\le m_1\}} + |u_\varepsilon|^{p-1}\chi_{\{|u_\varepsilon|>m_1\}} \\
&= |T_{m_1}(u_\varepsilon)|^{p-1} + |S_{m_1}(u_\varepsilon)|^{p-1},
\end{aligned}$$

and therefore, by the generalized Sobolev inequality (2.6), (4.58) and (4.59),

$$\begin{aligned}
\| |u_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} &= \| |T_{m_1}(u_\varepsilon)|^{p-1}|\|_{L^{\frac{N}{N-p},\infty}(\Omega)} + \| |S_{m_1}(u_\varepsilon)|^{p-1}|\|_{L^{\frac{N}{N-p},\infty}(\Omega)}, \\
&\le c\|\nabla T_{m_1}(u_\varepsilon)\|_{(L^p(\Omega))^N} + c \le c,
\end{aligned}$$

that is (4.5).

Now we prove Theorem 4.2 when assumption 2) in Theorem 3.1 is satisfied, i.e. $\gamma < \lambda = p-1$ and c_0 belonging to $L^{\frac{N}{p-1},\infty}(\Omega)$. We just observe that, under such assumptions, the proof made in the first case works exactly in the same way without any restriction on $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$, because γ is less than $p-1$.

Remark 4.3 In the proof of Theorem 4.2, when $\gamma = p-1$, we use a more general assumption on the summability of c_0 (4.6), that is $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$ with $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ small enough and not that $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < \infty$, with $\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)}$ small enough, as in the statement of Theorem 3.1 (see assumption 1)). This more restrictive assumption in Theorem 3.1 (which is an existence result) is due to our method wich uses the stability result of Theorem 5.1 in [GM] in which needs $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < \infty$ when $\gamma = p-1$.

4.2 Passing to the limit in the approximated problem

To conclude the proof of Theorem 3.1 we have to pass to the limit in the approximated problem (3.14). This is made exactly as in Section 5 of [GM] (cf. [BMMP3]). We repeat here the same arguments for the sake of completeness.

The solution u_ε of (3.14) satisfies

$$\begin{cases} -\operatorname{div}(a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)) = \Phi_\varepsilon - \operatorname{div}(g) + \operatorname{div}(F) & \text{in } \mathscr{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{4.60}$$

where

$$\begin{cases} \Phi_\varepsilon = f_\varepsilon - H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) - G_\varepsilon(x,u_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus, \\ \text{is bounded in } L^1(\Omega). \end{cases}$$

On the one hand using the growth condition (3.9) on H_ε and G_ε, Theorem 4.2 and the generalized Hølder inequality (2.4), we get

$$\|H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\|_{L^1(\Omega)} \le c \tag{4.61}$$

and

$$\|G_\varepsilon(x,u_\varepsilon)\|_{L^1(\Omega)} \le c. \tag{4.62}$$

On the other hand, using a $T_k(u_\varepsilon)$ as a test function in (4.60), since the norm of $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough, we easily obtain that for some M and L, we have

$$\int_\Omega |\nabla T_k(u_\varepsilon)|^p \le Mk + L, \tag{4.63}$$

for every $k > 0$ and every $\varepsilon > 0$.

Such an estimate and the growth condition (3.7) on K_ε allow us to use standard techniques (cf. [BMu, BG2, DMOP]) to extract a subsequence of u_ε still indexed by ε, such that

$$\begin{cases} u_\varepsilon \to u & \text{almost everywhere in } \Omega, \\ \nabla u_\varepsilon \to \nabla u & \text{almost everywhere in } \Omega, \\ \nabla T_k(u_\varepsilon) \to \nabla T_k(u) & \text{in } (L^p(\Omega))^N \text{ weakly,} \end{cases} \tag{4.64}$$

for every fixed $k \in \mathbb{N}$, where u is a function which is measurable on Ω, almost everywhere finite and such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k \in \mathbb{N}$, with a gradient ∇u as introduced in (2.18).

By (4.63) and by the Fatou lemma, we deduce that

$$\int_\Omega |\nabla T_k(u)|^p \le Mk + L,$$

and Lemma 4.1 gives

$$|u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega) \text{ and } |\nabla u|^{p-1} \in L^{\frac{N}{N-1},\infty}(\Omega).$$

From (4.64) and the definition (3.5) of H_ε, we deduce that

$$H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to H(x,u,\nabla u) \text{ almost everywhere in } \Omega. \tag{4.65}$$

Moreover, using the growth condition (3.9) on H_ε, Theorem 4.2 and the generalized Hølder inequality (2.4), we can prove that

$$H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \quad \text{is equi-integrable.}$$

Therefore the Vitali Theorem implies that

$$H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to H(x,u,\nabla u) \text{ in } L^1(\Omega) \text{ strongly.}$$

In a similar way we prove that

$$G_\varepsilon(x,u_\varepsilon) \to G(x,u) \text{ in } L^1(\Omega) \text{ strongly.}$$

Therefore the solution u_ε of (3.14) satisfies

$$\begin{cases} -\mathrm{div}(a(x,u_\varepsilon,\nabla u_\varepsilon) + K_\varepsilon(x,u_\varepsilon)) = f_\varepsilon - \Psi_\varepsilon - \mathrm{div}(g) + \mathrm{div}(F) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus & \text{in } \mathscr{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{4.66}$$

where u_ε satisfies (4.64) and

$$\Psi_\varepsilon = H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) + G_\varepsilon(x,u_\varepsilon) \to H(x,u,\nabla u) + G(x,u) \quad \text{in } L^1(\Omega) \text{ strongly,}$$

where $g \in (L^{p'}(\Omega))^N$ and where f_ε, $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ satisfy (3.1), (3.2) and (3.3).

Since u_ε is a weak solution of (4.66), it is also a renormalized solution of (4.66). Therefore we can apply the stability result in [GM], which is an extension of Theorem 3.2 proved in [DMOP] when $K(x,s) = 0$ (see also [MP]). It follows that u is a renormalized solution of

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u) + K(x,u)) + H(x,u,\nabla u) + G(x,u) = f - \mathrm{div}(g) + \mu_s^+ - \mu_s^- & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of Theorem 3.1 is completed. ■

Remark 4.4 Observe that we could prove an existence result in the case where $\gamma = p-1$, $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough and $\mu = f - \mathrm{div}(g)$ is a measure in $M_0(\Omega)$ (and not more a general measure). Indeed, under such assumptions, the a priori estimates given by Theorem 4.2 and the stability result used in Section 5 still hold true (see Remark 4.3 and also Remarks 4.2 and 4.7 in [GM])

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