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UNIQUENESS RESULTS FOR NONCOERCIVE NONLINEAR ELLIPTIC EQUATIONS WITH TWO LOWER ORDER TERMS

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Abstract. In the present paper we prove uniqueness results for weak solutions to a class of problems whose prototype is

$$\begin{cases} -\operatorname{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u) - \operatorname{div}(c(x)(1+|u|^2)^{(\tau+1)/2}) \\ + b(x)(1+|\nabla u|^2)^{(\sigma+1)/2} = f \text{ in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega) \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, p is a real number $\frac{2N}{N+1} < p < +\infty$, the coefficients $c(x)$ and $b(x)$ belong to suitable Lebesgue spaces, f is an element of the dual space $W^{-1,p'}(\Omega)$ and τ and σ are positive constants which belong to suitable intervals specified in Theorem 2.1, Theorem 2.2 and Theorem 2.3.

1. Introduction.

In the present paper we prove uniqueness results for weak solutions to a class of problems whose prototype is

$$\begin{cases} -\operatorname{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u) - \operatorname{div}(c(x)(1+|u|^2)^{(\tau+1)/2}) \\ \qquad + b(x)(1+|\nabla u|^2)^{(\sigma+1)/2} = f \text{ in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, p is a real number $\frac{2N}{N+1} < p < +\infty$, the coefficients $c(x)$ and $b(x)$ belong to suitable Lebesgue spaces, f is an element of the dual space $W^{-1,p'}(\Omega)$ and τ and σ are positive constants which belong to suitable intervals specified in Theorem 2.1, Theorem 2.2 and Theorem 2.3.

The main difficulty in dealing with existence or uniqueness of solutions to problem (1.1) is due to the presence of the two lower order terms, namely $b(x)(1+$

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$|\nabla u|^2)^{(\sigma+1)/2}$ and $\mathrm{div}(c(x)(1+|u|^2)^{(\tau+1)/2})$ which in general produces a lack of coercivity.

The linear case (i.e. $p = 2$) is investigated in [15] where existence and uniqueness results are given without any assumption on coercivity. As far as the nonlinear case is concerned, the existence of solutions to problem (1.1) has been proved in [18, 20] when f belongs to $W^{-1,p'}(\Omega)$, in [19] and in [23, 24] when f is a Radon measure with bounded total variation. It is worth noting that, when $\tau + 1 = \sigma + 1 = p - 1$, the existence of a solution is obtained under the assumption that the norm in appropriate spaces of one of the two coefficients c or b is small enough. When only one of the two terms $b(x)(1 + |\nabla u|^2)^{(\sigma+1)/2}$ or $\mathrm{div}(c(x)(1 + |u|^2)^{(\tau+1)/2})$ appears in problem (1.1), existence results are also established in various papers (see e.g. [7, 8, 9, 12, 21]), without any condition on the smallness on the data.

As far as the uniqueness of the solution to problem (1.1) is concerned, we recall that the case where $c \equiv 0$ is studied in [10], while in [8] uniqueness results for renormalized solutions in the case where f belongs to $L^1(\Omega)+W^{-1,p'}(\Omega)$ are proved. When $b \equiv 0$ and f is a function belonging to $L^1(\Omega)$, uniqueness of renormalized solutions to problem (1.1) is established in [7] and in [21] (in the linear case). Further uniqueness results can be found in [1, 2, 4, 6, 11, 14, 16, 17, 26] and in [13, 25, 27] for non-uniformly operators. Elliptic equations with a term of the type $b(x)|\nabla u|^p$ are studied in [3, 5].

The aim of the present paper is to study problem (1.1) where both terms $b(x)(1+|\nabla u|^2)^{(\sigma+1)/2}$ and $\mathrm{div}(c(x)(1 + |u|^2)^{(\tau+1)/2}$ appear. We will prove three uniqueness results, Theorems 2.1, 2.2 and 2.3 in which we do not make any assumption on the coercivity of the operator. We will prove such results under the assumptions of the existence result for the problem (1.1) proved in [20], that is $\tau+1 \le p-1$, $\sigma+1 \le p-1$ and that at least one of the norm of the coefficients c and b is small enough. We will prove different results according to the value of p, i.e. $2N/(N + 1) < p \le 2$ and $p \ge 2$. Such a difference is due to the principal part of the operator, which we consider. Actually we assume that when $p > 2$ the principal part $-\mathrm{div}(a(x, Du))$ is not degenerate, i.e. in the model case $-\mathrm{div}(a(x, \nabla u)) = -\mathrm{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u)$. But such an assumption is not required when $2N/(N + 1) < p \le 2$, that is for such values of p we prove uniqueness result for operator whose prototype is $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

We explicitly remark that Theorems 2.1, 2.2 and 2.3 coincide with the results proved in [10] in the case where $c \equiv 0$, but the techniques which we use in the present paper are quite different.

The proofs of our results are obtained in various steps. We firstly prove a priori estimate of the "reminder" $S_m(u - v)$ of the difference of two solutions $u - v$ to problem (1.1). Then we derive a "log–type estimate" (cf. e.g. [7, 12]) which implies by a contradiction argument that the two solutions u and v coincide. Actually such a log–type estimate is a quite different in the two cases; i.e. in the case where c is small enough and b is large and in the case where c is large and b is small enough.

2. Assumptions and main results.

In the present paper we consider a class of nonlinear elliptic problems of the type

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) - \mathrm{div}(\Phi(x, u)) + H(x, \nabla u) = f & \text{in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{2.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and p is a real number such that $\frac{2N}{N+1} < p < +\infty$. The function $a : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,\xi) \cdot \xi \geq \alpha |\xi|^p, \quad \alpha > 0, \tag{2.2}$$

$$|a(x,\xi)| \leq c[|\xi|^{p-1} + a_0(x)], \quad c > 0, \quad a_0(x) \in L^{p'}(\Omega), \quad a_0(x) \geq 0, \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. Moreover a is strongly monotone, that is a constant $\beta > 0$ exists such that

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq \begin{cases} \beta \dfrac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}} & \text{if } 1 \leq p \leq 2, \\ \beta |\xi - \eta|^2 (1 + |\xi| + |\eta|)^{p-2} & \text{if } p \geq 2, \end{cases} \tag{2.4}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.

We assume that $\Phi : \Omega \times \mathbb{R} \mapsto \mathbb{R}^N$ and $H : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions which satisfy the following "growth conditions"

$$|\Phi(x,s)| \leq c(x)(1 + |s|)^{p-1}, \quad c(x) \in L^t(\Omega), \ c(x) \geq 0, \tag{2.5}$$

with

$$\begin{cases} t \geq \dfrac{N}{p-1} & \text{if } p < N, \\ t > \dfrac{N}{N-1} & \text{if } p = N, \\ t \geq \dfrac{p}{p-1} & \text{if } p > N, \end{cases} \tag{2.6}$$

$$|H(x,\xi)| \leq b(x)(1 + |\xi|)^{p-1}, \quad b(x) \in L^r(\Omega), \quad b(x) \geq 0, \tag{2.7}$$

with

$$\begin{cases} r \geq N & \text{if } p < N, \\ r > N & \text{if } p = N, \\ r \geq p & \text{if } p > N, \end{cases} \tag{2.8}$$

for a. e. $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Moreover we assume that such functions are locally Lipschitz continuous with respect to the second variable, that is

$$|\Phi(x,s) - \Phi(x,z)| \leq c(x)(1 + |s| + |z|)^\tau |s - z|, \quad \tau \geq 0, \tag{2.9}$$

$$|H(x,\xi) - H(x,\eta)| \leq b(x)(1 + |\xi| + |\eta|)^\sigma |\xi - \eta|, \quad \sigma \geq 0, \tag{2.10}$$

for almost every $x \in \Omega$, for every $s, z \in \mathbb{R}$ and for every $\xi, \eta \in \mathbb{R}^N$.

Finally we assume that

$$f \in W^{-1,p'}(\Omega). \tag{2.11}$$

In the present paper we will prove uniqueness results for weak solutions to problem (2.1), i.e. for functions u which satisfy the following condition

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u) \cdot \nabla \varphi + \int_\Omega \Phi(x,u) \cdot \nabla \varphi \\ \qquad \displaystyle + \int_\Omega H(x,\nabla u)\varphi = < f, \varphi >_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}, \quad \forall \varphi \in W_0^{1,p}(\Omega). \end{cases} \tag{2.12}$$

The existence of such a solution is proved in [15] in the linear case (i.e. $p = 2$) and in [20] in the general case (cfr. [22] for a different proof) under the assumptions (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.11) and the assumption that one of the norm of coefficients b and c is small enough.

Remark 1. Let us compare the assumption (2.5) on the growth condition and the assumption (2.9) on the locally Lipschitz continuity made on Φ. Observe that assumption (2.9) implies a growth condition on Φ, i.e.

$$|\Phi(x,s)| \leq c(x)(1+|s|)^{\tau+1} + |\Phi(x,0)|. \tag{2.13}$$

But condition (2.13) can be more restrictive than (2.5) depending on the value of τ (for example when $\tau + 1 < p - 1$). This is the reason for which we assume both (2.5) and (2.9).

A similar comparison holds true for the assumption (2.7) on the growth condition and the assumption (2.10) on the locally Lipschitz condition made on H. Actually assumption (2.10) implies a growth condition on H which can be more restrictive than (2.7) depending on the value of σ (for example when $\sigma + 1 < p - 1$).

Remark 2. The model function $a(x,\xi)$ which satisfies assumptions (2.3) and (2.4) is

$$a(x,\xi) = \begin{cases} a(x)|\xi|^{p-2}\xi & \text{if } 1 < p \leq 2, \\ a(x)(1+|\xi|^2)^{(p-2)/2}\xi & \text{if } p > 2, \end{cases}$$

where $a(x)$ is a function belonging to $L^\infty(\Omega)$ such that $a(x) \geq \alpha > 0$.

Examples of functions $\Phi(x,s)$ and $H(x,\xi)$ are given by

$$\Phi(x,s) = c(x)(1+|s|)^\gamma, \qquad \text{and} \qquad H(x,\xi) = b(x)(1+|\xi|^2)^\lambda,$$

where $c(x) \in L^t(\Omega)$, $t \geq N/(p-1)$ and $b(x) \in L^r(\Omega)$, $r \geq N$, $\gamma = \min(p-1, \tau+1)$, $\lambda = \min(\sigma+1, p-1)/2$.

We will prove three uniqueness results stated in Theorem 2.1, Theorem 2.2 and Theorem 2.3 which correspond to the case $\frac{2N}{N+1} < p \leq 2$ and $N \geq 3$, to the case $N \geq p \geq 2$ and $N \geq 3$, and to the case $\frac{2N}{N+1} = \frac{4}{3} < p \leq 2$ and $N = 2$. Such cases are due to the assumption (2.3) on the operator a, which presents a "degeneracy" in the case $1 \leq p < 2$ and a "non–degeneracy" in the case $p \geq 2$. The case where $p > N$ is also considered (see Remark 7 below)

We begin by stating the uniqueness result in the case $\frac{2N}{N+1} < p \leq 2$ and $N \geq 3$.

Theorem 2.1. *Let $\frac{2N}{N+1} < p \leq 2$ and $N \geq 3$. We assume that (2.2)-(2.11) are satisfied with*

$$\|c\|_{L^t(\Omega)} \text{ or } \|b\|_{L^r(\Omega)} \text{ small enough,} \tag{2.14}$$

$$\frac{Np}{Np-2N+p} \leq t < +\infty, \tag{2.15}$$

$$\frac{Np}{Np-2N+p} \leq r < +\infty, \tag{2.16}$$

$$0 \leq \tau \leq \tau^*(N,p,t) = \frac{Np}{N-p}\Big(1 - \frac{2}{p} + \frac{1}{N} - \frac{1}{t}\Big), \tag{2.17}$$

$$0 \leq \sigma \leq \sigma^*(N,p,r) = p\Big(1 - \frac{2}{p} + \frac{1}{N} - \frac{1}{r}\Big). \tag{2.18}$$

If u and v are two weak solutions to problem (2.1), then $u \equiv v$ a.e. in Ω.

Remark 3. Observe that the bounds on p and N imply that $1 < p < N$. Moreover the assumption $p > \frac{2N}{N+1}$ implies $\frac{Np}{Np-2N+p} > 0$. Finally observe that the assumptions (2.15) on t and (2.16) on r imply $\frac{1}{t} \le 1 - \frac{2}{p} + \frac{1}{N}$ and $\frac{1}{r} \le 1 - \frac{2}{p} + \frac{1}{N}$. Therefore $\tau^*(N,t,p)$ and $\sigma^*(N,t,p)$ are non negative constants.

Remark 4. In [20] it is proved the existence of a weak solution to problem (2.1) under the assumptions (2.2)–(2.8), (2.11) and (2.14). Moreover in the proof of such a result it is performed an a priori estimate for solutions to problem (2.1), which we will use in the proof of our uniqueness theorems and which we will state below:

Let u be any solution to problem (2.1) for a fixed datum $f \in W^{-1,p'}(\Omega)$ and a fixed coefficient $c \in L^t(\Omega)$ where t satisfies (2.6). There exist two constants $C > 0$ and $\eta > 0$ which depend on $|\Omega|$, p, N, α, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$ (but not on $\|b\|_{L^r(\Omega)}$ with r such that (2.8) is satisfied) such that

$$\text{if } \|b\|_{L^r(\Omega)} \le \eta, \quad \text{then} \quad \|u\|_{W_0^{1,p}(\Omega)} \le C. \tag{2.19}$$

Let u be any solution to problem (2.1) for a fixed datum $f \in W^{-1,p'}(\Omega)$ and a fixed coefficient $b \in L^r(\Omega)$. There exist two constants $C > 0$ and $\eta' > 0$ which depend on $|\Omega|$, p, N, α, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$ (but not on $\|c\|_{L^t(\Omega)}$) such that

$$\text{if } \|c\|_{L^t(\Omega)} \le \eta', \quad \text{then} \quad \|u\|_{W_0^{1,p}(\Omega)} \le C. \tag{2.20}$$

Now we state our uniqueness result in the case where $N \ge p \ge 2$ and $N \ge 3$.

Theorem 2.2. *Let $N \ge p \ge 2$ and $N \ge 3$. We assume that (2.2)-(2.11) are satisfied with*

$$\|c\|_{L^t(\Omega)} \text{ or } \|b\|_{L^r(\Omega)} \text{ small enough}, \tag{2.21}$$

$$N \le t < +\infty, \tag{2.22}$$

$$N \le r < +\infty, \tag{2.23}$$

$$0 \le \tau \le \tau^{**}(N,p,t) = \begin{cases} \dfrac{Np}{N-p}\Big(\dfrac{1}{N} - \dfrac{1}{t}\Big) & \text{if } p < N \\ 0 & \text{if } p = N = t, \\ \text{any } q,\ 0 < q < +\infty & \text{if } p = N \text{ and } N < t. \end{cases} \tag{2.24}$$

$$0 \le \sigma \le \sigma^{**}(N,p,r) = p\Big(\frac{1}{N} - \frac{1}{r}\Big). \tag{2.25}$$

If u and v are two weak solutions of problem (2.1), then $u \equiv v$ a.e. in Ω.

Remark 5. Observe that the bounds on t and r imply that $\tau^{**}(N,p,t)$ and $\sigma^{**}(N,p,r)$ are positive constants (except the case $t = N$ where $\tau^{**}(N,p,t) = 0$ and $r = N$ where $\sigma^{**} = 0$).

Remark 6. In [20] it is proved the existence of a weak solution to the problem (2.1) under the assumptions (2.2), (2.3), (2.4), (2.5), (2.7), (2.8), (2.11) and (2.21). Moreover in the proof of such a result it is performed an a priori estimate for solutions to problem (2.1), which we will use in the proof of our uniqueness results and which we stated in Remark 4 above.

Remark 7. Observe that the assumptions made on p in Theorems 2.1 and 2.2 imply $p \le N$. Actually we can prove, by adapting the proof of Theorem 2.2, an uniqueness result for weak solutions to problem (2.1) under the assumptions $p > N$, (2.2)-(2.8), (2.10), (2.11) and the assumption on local Lipschitz continuity on Φ (instead of the more restrictive global condition (2.9)), i.e.

$$\forall K > 0,\ \exists C > 0, \quad |\Phi(x,s) - \Phi(x,z)| \le Cb(x)|s-z|,$$

forall $s, z \in \mathbb{R}$, $|s| \le K$, $|z| \le K$, a.e. in Ω, with $\tau \le \tau^{**}(N,p,t) =$ any q, $0 < q < +\infty$.

Now we state our uniquenness result in the case where $N = 2$.

Theorem 2.3. *Let $N = 2$. We assume that (2.2)-(2.11) are satisfied with*

$$\|c\|_{L^t(\Omega)} \text{ or } \|b\|_{L^r(\Omega)} \text{ small enough.} \tag{2.26}$$

Moreover we assume that

(i) *if* $\frac{4}{3} = \frac{2N}{N+1} < p < 2$ *then*

$$2 \le t < +\infty, \quad 2 \le r < +\infty,$$
$$0 \le \tau \le \tau^*(2,p,t) = \frac{2p}{2-p}\Big(\frac{1}{2} - \frac{1}{t}\Big), \quad 0 \le \sigma \le \sigma^*(2,p,r) = p\Big(\frac{1}{2} - \frac{1}{r}\Big);$$

(ii) *if* $p = 2$, *then*

$$2 < t < +\infty, \quad 2 < r < +\infty,$$
$$0 \le \tau < \tau^{**}(2,2,t) = 2, \quad 0 \le \sigma < \sigma^{**}(2,2,r) = 1 - \frac{2}{r};$$

If u and v are two weak solutions of problem (2.1), then $u \equiv v$ almost everywhere in Ω.

Remark 8. We explicitly remark that Theorems 2.1, 2.2 and 2.3 in the case where the problem (2.1) does not contain the term $-\mathrm{div}(\Phi(x,u))$, i.e. $\Phi \equiv 0$, coincide with the uniqueness results proved in [10].

Observe that we prove Theorem 2.3 under the assumption $p \le 2$. The case where $p > 2$ corresponds to the case $p > N$ (see Remark 7 above).

3. Proofs of Theorems.

3.1. **Proof of Theorem 2.1.** The proof is performed in three steps which correspond to the case where $\|c\|_{L^t(\Omega)}$ and $\|b\|_{L^r(\Omega)}$ are small enough, the case where $\|c\|_{L^t(\Omega)}$ is small enough and $\|b\|_{L^r(\Omega)}$ is large and the case where $\|c\|_{L^t(\Omega)}$ is large and $\|b\|_{L^r(\Omega)}$ is small enough. In any case we begin the proof by deriving an a priori estimate for $S_m(u-v)$, the "reminder" of the difference of $u-v$ and then we argue by contradiction. In the first case the conclusion that $u \equiv v$ a.e. in Ω is a simple consequence of the a priori estimate for $S_m(u-v)$, while in the second and third cases we prove a "log–type" estimate for the difference $u - v$ which implies that $u \equiv v$ a.e. in Ω (cf. [12, 7]).

Step 1. *The case where $\|c\|_{L^t(\Omega)}$ and $\|b\|_{L^r(\Omega)}$ are small enough.*

For $m > 0$, we denote by $S_m : \mathbb{R} \mapsto \mathbb{R}$ the function

$$S_m(s) = \begin{cases} (|s| - m)\mathrm{sign}(s) & |s| > m, \\ 0 & |s| \le m, \end{cases} \tag{3.1}$$

for any $s \in \mathbb{R}$.

We consider $S_m(u-v)$ as test function in (2.12) satisfied by u and (2.12) satisfied by v. Then we subtract the results and we obtain

$$\int_\Omega (a(x,\nabla u) - a(x,\nabla v)) \cdot \nabla S_m(u-v) + \int_\Omega (\Phi(x,u) - \Phi(x,v)) \cdot \nabla S_m(u-v) + \int_\Omega (H(x,\nabla u) - H(x,\nabla v)) S_m(u-v) = 0. \quad (3.2)$$

By the monotonicity assumption (2.4) on a, and the assumptions of locally Lipschitz continuity (2.9) on Φ and (2.10) on H, we get

$$\beta \int_{\{|u-v|>m\}} \frac{|\nabla(u-v)|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \le \int_{\{|u-v|>m\}} c(x)(1+|u|+|v|)^\tau |u-v||\nabla S_m(u-v)| + \int_{\{|u-v|>m\}} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla(u-v)||S_m(u-v)|. \quad (3.3)$$

As in [8], we define the following set Z. As $|\Omega|$ is finite, the set of the constants k such that $|\{x \in \Omega : |(u-v)(x)| = k\}| > 0$ is at most countable. Let Z^c be the (countable) union of all those sets. Its complementary $Z = \Omega \setminus Z^c$ is therefore the union of the sets such that $|\{x \in \Omega : |(u-v)(x)| = k\}| = 0$. Since for every k,

$$\nabla(u-v) = 0 \qquad \text{a.e. on } \{x \in \Omega,\ |(u-v)(x)| = k\},$$

and since Z^c is at most a countable union, we obtain that

$$\nabla u - \nabla v = 0 \quad \text{a.e. on } Z^c. \quad (3.4)$$

We deduce by (3.3) and (3.4) that

$$\beta \int_{\{|u-v|>m\}} \frac{|\nabla(u-v)|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \le I_1 + I_2, \quad (3.5)$$

where

$$I_1 = \int_{\{|u-v|>m\}\cap Z} c(x)(1+|u|+|v|)^\tau |u-v||\nabla S_m(u-v)|, \quad (3.6)$$

$$I_2 = \int_{\{|u-v|>m\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla(u-v)||S_m(u-v)|. \quad (3.7)$$

Now we estimate I_1 and I_2. As far as I_1 is concerned, we have

$$\begin{aligned} I_1 &= \int_{\{|u-v|>m\}\cap Z} c(x)(1+|u|+|v|)^\tau |u-v||\nabla S_m(u-v)| \\ &\le \int_{\{|u-v|>m\}\cap Z} c(x)(1+|u|+|v|)^\tau |S_m(u-v)||\nabla S_m(u-v)| \\ &\quad + m \int_{\{|u-v|>m\}\cap Z} c(x)(1+|u|+|v|)^\tau |\nabla S_m(u-v)|. \end{aligned} \quad (3.8)$$

Since $p < N$, assumption (2.17) on τ is equivalent to

$$\frac{1}{t} + \frac{\tau}{p^*} + \frac{1}{p^*} + \frac{1}{p} \le 1,$$

where $p^* = Np/(N-p)$. Therefore by using Hölder's inequality and Sobolev's embedding Theorem in (3.8), we obtain

$$\begin{aligned} I_1 \le &\|c\|_{L^t(\{|u-v|>m\}\cap Z)} \|1+|u|+|v|\|^\tau_{L^{p^*}(\Omega)} \|\nabla S_m(u-v)\|_{(L^p(\Omega))^N} \\ &\times \left(S_{N,p} \|\nabla S_m(u-v)\|_{(L^p(\Omega))^N} + m|\Omega|^{1-1/p^*} \right) \end{aligned} \quad (3.9)$$

where $S_{N,p}$ denotes the best constant in the embedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$.

Now we evaluate I_2. Since $p < N$, assumption (2.18) on σ is equivalent to

$$\frac{1}{r} + \frac{\sigma}{p} + \frac{1}{p^*} + \frac{1}{p} \leq 1.$$

Therefore we can apply Hölder's inequality in (3.7) and by using Sobolev's embedding Theorem, we have

$$\begin{aligned} I_2 =& \int_{\{|u-v|>m\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^{\sigma}|\nabla(u-v)||S_m(u-v)| \\ \leq& \|b\|_{L^r(\{|u-v|>m\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} S_{N,p}\|\nabla S_m(u-v)\|^2_{(L^p(\Omega))^N}. \end{aligned} \tag{3.10}$$

Combining (3.5), (3.9) and (3.10), we get

$$\begin{aligned} \int_{\{|u-v|>m\}} \frac{|\nabla(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}} \leq& \frac{S_{N,p}}{\beta}\|c\|_{L^t(\{|u-v|>m\}\cap Z)}\|1+|u|+|v|\|^{\tau}_{L^{p^*}(\Omega)} \\ &\times \|\nabla S_m(u-v)\|^2_{(L^p(\Omega))^N} \\ &+ \frac{S_{N,p}}{\beta}\|b\|_{L^r(\{|u-v|>m\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)}\|\nabla S_m(u-v)\|^2_{(L^p(\Omega))^N} \\ &+ \frac{m}{\beta}|\Omega|^{1-1/p^*}\|c\|_{L^t(\{|u-v|>m\}\cap Z)}\|1+|u|+|v|\|^{\tau}_{L^{p^*}(\Omega)}\|\nabla S_m(u-v)\|_{(L^p(\Omega))^N}. \end{aligned} \tag{3.11}$$

On the other hand by Hölder's inequality, since $p \leq 2$, we have

$$\begin{aligned} &\int_{\{|u-v|>m\}} |\nabla(u-v)|^p \\ \leq& \left(\int_{\{|u-v|>m\}} \frac{|\nabla(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\right)^{p/2} \left(\int_{\{|u-v|>m\}} (|\nabla u|+|\nabla v|)^p\right)^{1-p/2}. \end{aligned} \tag{3.12}$$

Combining (3.12) and (3.11), we obtain

$$\begin{aligned} \|\nabla S_m(u-v)\|^p_{(L^p(\Omega))^N} \leq& \||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)} \\ &\times \frac{1}{\beta^{p/2}}\Big\{ S^{p/2}_{N,p}\|c\|^{p/2}_{L^t(\{|u-v|>m\}\cap Z)}\|1+|u|+|v|\|^{\tau p/2}_{L^{p^*}(\Omega)}\|\nabla S_m(u-v)\|^p_{(L^p(\Omega))^N} \\ &+ S^{p/2}_{N,p}\|b\|^{p/2}_{L^r(\{|u-v|>m\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\sigma p/2}_{L^p(\Omega)}\|\nabla S_m(u-v)\|^p_{(L^p(\Omega))^N} \\ &+ m^{p/2}|\Omega|^{(1-1/p^*)p/2}\|c\|^{p/2}_{L^t(\{|u-v|>m\}\cap Z)}\|1+|u|+|v|\|^{\tau p/2}_{L^{p^*}(\Omega)}\|\nabla S_m(u-v)\|^{p/2}_{(L^p(\Omega))^N}\Big\}. \end{aligned} \tag{3.13}$$

Now we argue by contradiction, i.e. let us assume that

$$\big|\{x \in \Omega : |u(x)-v(x)| > 0\}\big| > 0.$$

Moreover assume that

$$\|c\|_{L^t(\{|u-v|>0\}\cap Z)} \text{ and } \|b\|_{L^r(\{|u-v|>0\}\cap Z)} \text{ are small enough,}$$

i.e.

$$\begin{cases} \dfrac{S^{p/2}_{N,p}}{\beta^{p/2}}\|c\|^{p/2}_{L^t(\{|u-v|>0\}\cap Z)}\|1+|u|+|v|\|^{\tau p/2}_{L^{p^*}(\Omega)}\||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)} \leq \dfrac{1}{4}, \\ \dfrac{S^{p/2}_{N,p}}{\beta^{p/2}}\|b\|^{p/2}_{L^r(\{|u-v|>0\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\sigma p/2}_{L^p(\Omega)}\||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)} \leq \dfrac{1}{4}. \end{cases} \tag{3.14}$$

Then we can choose $m = 0$ in (3.13) and we get

$$\|\nabla(u-v)\|^p_{(L^p(\Omega))^N} \le 0,$$

which gives a contradiction. Therefore we conclude that if (3.14) holds true, then

$$\big|\{x \in \Omega : \ |u(x)-v(x)| > 0\}\big| = 0,$$

i.e. $u = v$ almost everywhere in Ω.

Step 2. *The case where $\|c\|_{L^t(\Omega)}$ is large and $\|b\|_{L^r(\Omega)}$ is small enough.*

We begin by observing that the proof of estimate (3.13) in the previous Step is made without any assumption on the smallness of the norms $\|b\|^{p/2}_{L^r(\Omega)}$ or $\|c\|^{p/2}_{L^t(\Omega)}$.

Now we argue by contradiction, i.e. let us assume that

$$\big|\{x \in \Omega : \ |u(x)-v(x)| > 0\}\big| > 0.$$

Moreover assume that $\|b\|_{L^r(\Omega)}$ is small enough, that is

$$\|b\|^{p/2}_{L^r(\Omega)} \le \eta, \tag{3.15}$$

where $\eta > 0$ is the constant defined in Remark 4. By (2.19) in Remark 4, the terms $\|1+|u|+|v|\|^{\tau p/2}_{L^{p^*}(\Omega)}\||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)}$ and $\|1+|\nabla u|+|\nabla v|\|^{\sigma p/2}_{L^p(\Omega)}\||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)}$ are bounded by a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b. Therefore by (3.13), we obtain

$$\begin{aligned}\|\nabla S_m(u-v)\|^p_{(L^p(\Omega))^N} \le{}& C_0\|c\|^{p/2}_{L^t(\{|u-v|>m\}\cap Z)}\|\nabla S_m(u-v)\|^p_{(L^p(\Omega))^N}\\ &+ C_0\|b\|^{p/2}_{L^r(\{|u-v|>m\}\cap Z)}\|\nabla S_m(u-v)\|^p_{(L^p(\Omega))^N}\\ &+ C_0 m^{p/2}\|c\|^{p/2}_{L^t(\{|u-v|>m\}\cap Z)}\|\nabla S_m(u-v)\|^{p/2}_{(L^p(\Omega))^N},\end{aligned} \tag{3.16}$$

where C_0 is a positive constant which depends only on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

In this Step we assume that (3.14) does not hold and that the following conditions are satisfied

$$C_0\|c\|^{p/2}_{L^t(\{|u-v|>0\}\cap Z)} > \frac{1}{4} \quad \text{and} \quad \|b\|^{p/2}_{L^r(\Omega)} \le B, \tag{3.17}$$

where B is a constant small enough, i.e.

$$B \le \min\left\{\eta, \frac{1}{4C_0}\right\}, \tag{3.18}$$

and which will be better specified later.

Now let us consider the function $G : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$G(m) = C_0\|c\|^{p/2}_{L^t(\{|u-v|>m\}\cap Z)}.$$

It is continuous, decreasing, it tends to zero as m goes to infinity and, since we assume (3.17), it verifies $G(0) > 1/4$. Therefore there exists $m = m_1 > 0$ such that

$$G(m_1) = C_0\|c\|^{p/2}_{L^t(\{|u-v|>m_1\}\cap Z)} = \frac{1}{4}.$$

By (3.18) we have $C_0\|b\|^{p/2}_{L^r(\Omega)} \le C_0 B \le 1/4$ and then by (3.16) we get

$$\|\nabla S_{m_1}(u-v)\|^p_{(L^p(\Omega))^N} \le \frac{1}{2}{m_1}^{p/2}\|\nabla S_{m_1}(u-v)\|^{p/2}_{(L^p(\Omega))^N},$$

or, equivalently,

$$\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N} \leq \left(\frac{1}{2}\right)^{2/p} m_1. \tag{3.19}$$

We now derive a technical "log-type estimate" on $u-v$. Denote by $\varphi : \mathbb{R} \mapsto \mathbb{R}$ the function defined by

$$\varphi(w) = \int_0^w \frac{ds}{(M+|s|)^2}, \qquad \forall w \in \mathbb{R}, \tag{3.20}$$

where $M > 0$ is a constant which will be specified later.

Since $T_{m_1}(u-v)$ belongs to $W_0^{1,p}(\Omega)$ and φ is a Lipschitz function such that $\varphi(0) = 0$, then the function $\varphi(T_{m_1}(u-v))$ belongs to $W_0^{1,p}(\Omega)$. Moreover, by the definition of φ, we have

$$\left|\varphi(T_{m_1}(u-v))\right| \leq \int_0^{m_1} \frac{ds}{(M+|s|)^2} = \frac{m_1}{M(M+m_1)}. \tag{3.21}$$

Let us choose $\varphi(T_{m_1}(u-v))$ as test function in the equality (2.12) satisfied by u and in the equality (2.12) satisfied by v. By subtracting the two results, we get

$$\begin{aligned}\int_{\{|u-v|<m_1\}} (a(x,\nabla u) - a(x,\nabla v)) \cdot \nabla(u-v)\varphi'(T_{m_1}(u-v)) \\ + \int_{\{|u-v|<m_1\}} (\Phi(x,u) - \Phi(x,v)) \cdot \nabla(u-v)\varphi'(T_{m_1}(u-v)) \\ + \int_\Omega (H(x,\nabla u) - H(x,\nabla v))\varphi(T_{m_1}(u-v)) = 0.\end{aligned} \tag{3.22}$$

By the monotonicity assumption (2.4) on a, the assumptions of locally Lipschitz continuity (2.9) on Φ and (2.10) on H and since $\varphi'(w) = \frac{1}{(M+|w|)^2}$, we get

$$\begin{aligned}\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \frac{1}{(|\nabla u| + |\nabla v|)^{2-p}} \\ \leq \int_{\{|u-v|<m_1\}} c(x)(1+|u|+|v|)^\tau |u-v| \frac{|\nabla(u-v)|}{(M+|u-v|)^2} \\ + \int_\Omega b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla u - \nabla v||\varphi(T_{m_1}(u-v))|.\end{aligned}$$

As in Step 1 we define the set Z. Since by (3.4), $\nabla u - \nabla v = 0$ almost everywhere on Z^c, we deduce

$$\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \frac{1}{(|\nabla u| + |\nabla v|)^{2-p}} \leq J_1 + J_2, \tag{3.23}$$

where

$$J_1 = \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau |u-v| \frac{|\nabla(u-v)|}{(M+|u-v|)^2},$$

$$J_2 = \int_Z b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla u - \nabla v||\varphi(T_{m_1}(u-v))|.$$

Now we will estimate J_1 and J_2. We begin by evaluating J_1.

Since $\left|\frac{T_{m_1}(u-v)}{M+|T_{m_1}(u-v)|}\right| \leq 1$, we have

$$\begin{aligned} J_1 &= \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau |T_{m_1}(u-v)| \frac{|\nabla(u-v)|}{(M+|u-v|)^2} \\ &\leq \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau \frac{|\nabla(u-v)|}{(M+|u-v|)}. \end{aligned} \tag{3.24}$$

Since $p<N$, assumption (2.17) on τ implies that

$$\frac{1}{t}+\frac{\tau}{p^*}+\frac{1}{p}<1.$$

Therefore we can apply Hölder's inequality in (3.24) and we get

$$\begin{aligned} J_1 \leq\ & |\Omega|^{1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{p}} \|c\|_{L^t(\Omega)} \|1+|u|+|v|\|^\tau_{L^{p^*}(\Omega)} \\ & \times \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p}\Big)^{1/p}. \end{aligned} \tag{3.25}$$

Now we evaluate J_2. From (3.21), since $M+m_1>M+|u-v|$ almost everywhere on the set $\{|u-v|<m_1\}\cap Z$, we have

$$\begin{aligned} J_2 &\leq \frac{m_1}{M(M+m_1)} \int_{\{|u-v|<m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla(u-v)| \\ &\quad + \frac{m_1}{M(M+m_1)} \int_{\{|u-v|>m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla S_{m_1}(u-v)| \\ &\leq \frac{m_1}{M} \int_{\{|u-v|<m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma \frac{|\nabla(u-v)|}{M+|u-v|} \\ &\quad + \frac{m_1}{M(M+m_1)} \int_{\{|u-v|>m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla S_{m_1}(u-v)|. \end{aligned} \tag{3.26}$$

Moreover, since $p<N$, assumption (2.18) on σ implies that

$$\frac{1}{r}+\frac{\sigma}{p}+\frac{1}{p}<1.$$

Hence Hölder's inequality in (3.26) leads to

$$\begin{aligned} J_2 \leq & \frac{m_1}{M} |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\{|u-v|<m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)} \\ & \times \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p}\Big)^{1/p} \\ & + \frac{m_1|\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}}}{M(M+m_1)} \|b\|_{L^r(\{|u-v|>m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)} \\ & \times \|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}. \end{aligned} \tag{3.27}$$

Therefore combining (3.23), (3.25) and (3.27) leads to

$$\begin{aligned}
&\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \frac{1}{(|\nabla u|+|\nabla v|)^{2-p}} \\
&\le |\Omega|^{1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{p}} \|c\|_{L^t(\Omega)} \|1+|u|+|v|\|^{\tau}_{L^{p^*}(\Omega)} \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \Big)^{1/p} \\
&\quad + \frac{m_1}{M} |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\{|u-v|<m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \\
&\quad \times \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \Big)^{1/p} \\
&\quad + \frac{m_1}{M(M+m_1)} |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\{|u-v|>m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \\
&\quad \times \|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}.
\end{aligned} \tag{3.28}$$

On the other hand from Hölder's inequality, since $p \le 2$, it follows that

$$\begin{aligned}
\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} &\le \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \frac{1}{(|\nabla u|+|\nabla v|)^{2-p}} \right)^{p/2} \\
&\quad \times \left(\int_{\Omega} (|\nabla u|+|\nabla v|)^p \right)^{(2-p)/2}.
\end{aligned} \tag{3.29}$$

Gathering (3.28) and (3.29), we obtain

$$\begin{aligned}
\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} &\le \frac{1}{\beta^{p/2}} \||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)} \\
&\times \Bigg\{ |\Omega|^{(1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{p})\frac{p}{2}} \|c\|^{p/2}_{L^t(\Omega)} \|1+|u|+|v|\|^{\tau p/2}_{L^{p^*}(\Omega)} \\
&\qquad \times \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \Big)^{1/2} \\
&\qquad + \frac{m_1^{p/2}}{M^{p/2}} |\Omega|^{(1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p})\frac{p}{2}} \|b\|^{p/2}_{L^r(\{|u-v|<m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^{\sigma p/2}_{L^p(\Omega)} \\
&\qquad \times \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \Big)^{1/2} \\
&\qquad + \left(\frac{m_1}{M(M+m_1)} \right)^{p/2} |\Omega|^{(1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p})\frac{p}{2}} \|b\|^{p/2}_{L^r(\{|u-v|>m_1\}\cap Z)} \\
&\qquad \times \|1+|\nabla u|+|\nabla v|\|^{\sigma p/2}_{L^p(\Omega)} \|\nabla S_{m_1}(u-v)\|^{p/2}_{(L^p(\Omega))^N} \Bigg\}.
\end{aligned} \tag{3.30}$$

Now we observe that, by (2.19) in Remark 4 and by (3.17), since we chose in (3.18) B in such a way that $B \le \eta$, the terms $\|1+|u|+|v|\|^{\tau p/2}_{L^{p^*}(\Omega)} \||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)}$ and $\|1+|\nabla u|+|\nabla v|\|^{\sigma p/2}_{L^p(\Omega)} \||\nabla u|+|\nabla v|\|^{(2-p)p/2}_{L^p(\Omega)}$ are bounded by a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

Therefore by (3.30) we get

$$\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \le C_1 \|c\|^{p/2}_{L^t(\Omega)} \Big(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \Big)^{1/2}$$
$$+ C_1 \frac{m_1^{p/2}}{M^{p/2}} \|b\|^{p/2}_{L^r(\{|u-v|<m_1\}\cap Z)} \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \right)^{1/2}$$
$$+ C_1 \frac{m_1^{p/2}}{M^{p/2}(M+m_1)^{p/2}} \|b\|^{p/2}_{L^r(\{|u-v|>m_1\}\cap Z)} \|\nabla S_{m_1}(u-v)\|^{p/2}_{(L^p(\Omega))^N},$$

where C_1 is a positive constant which depends on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

Moreover by Young's inequality, estimate (3.19) of $\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}$, since $(M+m_1)^{p/2} \ge M^{p/2}$, we get

$$\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} \le C_2 \|c\|^p_{L^t(\Omega)} + C_2 \frac{m_1^p}{M^p} \|b\|^p_{L^r(\{|u-v|<m_1\}\cap Z)}$$
$$+ C_2 \frac{m_1^p}{M^p} \|b\|^{p/2}_{L^r(\{|u-v|>m_1\}\cap Z)}, \tag{3.31}$$

where C_2 is a positive constant which depends on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

Now let us observe that the equiintegrability of c implies the existence of a positive constant δ_0, depending on c and on C_0 (but not on b), such that

$$|\{|u-v| > m_1\} \cap Z| > \delta_0.$$

Therefore we have

$$\delta_0 \le \delta = \big|(Z \cap \{|u-v| > m_1\})\big| \le \big|\{|u-v| > m_1\}\big|.$$

By using Poincaré's inequality, we get

$$\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M+|u-v|)^p} = \int_\Omega \frac{|\nabla T_{m_1}(u-v)|^p}{(M+|u-v|)^p}$$
$$= \int_\Omega \left| \nabla \ln\left(1 + \frac{|T_{m_1}(u-v)|}{M}\right)\right|^p \ge C(|\Omega|, p) \int_\Omega \left| \ln\left(1 + \frac{|T_{m_1}(u-v)|}{M}\right)\right|^p$$
$$\ge C(|\Omega|, p) \int_{\{|u-v|>m_1\}} \left| \ln\left(1 + \frac{|T_{m_1}(u-v)|}{M}\right)\right|^p \ge C(|\Omega|, p)\delta_0 \left[\ln\left(1 + \frac{m_1}{M}\right)\right]^p. \tag{3.32}$$

Combining (3.31) and (3.32) we have

$$C(|\Omega|, p)\delta_0 \left[\ln\left(1 + \frac{m_1}{M}\right)\right]^p \le C_2 \|c\|^p_{L^t(\Omega)} + C_2 \frac{m_1^p}{M^p} \|b\|^p_{L^r(\{|u-v|<m_1\}\cap Z)}$$
$$+ C_2 \frac{m_1^p}{M^p} \|b\|^{p/2}_{L^r(\{|u-v|>m_1\}\cap Z)}. \tag{3.33}$$

Now let us observe that, since M is a positive constant to be specified, we can denote

$$M = m_1 \varepsilon,$$

where $\varepsilon > 0$ will be chosen later. Therefore by (3.33), we get

$$C(|\Omega|, p)\delta_0 \left[\ln\left(1 + \frac{1}{\varepsilon}\right)\right]^p \le C_2 \|c\|^p_{L^t(\Omega)} + C_2 \frac{1}{\varepsilon^p} \|b\|^p_{L^r(\Omega)} + C_2 \frac{1}{\varepsilon^p} \|b\|^{p/2}_{L^r(\Omega)},$$

and since, by (3.17), $\|b\|_{L^r(\Omega)}^{p/2} \le B$ then we have

$$\delta_0 \left[\ln \left(1 + \frac{1}{\varepsilon} \right) \right]^p \le C_3 + \frac{C_3}{\varepsilon^p} B^2 + \frac{C_3}{\varepsilon^p} B, \tag{3.34}$$

where C_3 is a positive constant which depends on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$ (but not on b). Now we choose $\varepsilon = \varepsilon_1 > 0$ in such a way that

$$\delta_0 \left[\ln \left(1 + \frac{1}{\varepsilon_1} \right) \right]^p \ge 2C_3.$$

It is worth noting that ε_1 does not depend on b, on u and on v. Therefore (3.34) implies

$$2C_3 \le C_3 + \frac{C_3}{\varepsilon_1^p} B^2 + \frac{C_3}{\varepsilon_1^p} B$$

or, equivalently,

$$1 \le \frac{B^2}{\varepsilon_1^p} + \frac{B}{\varepsilon_1^p}. \tag{3.35}$$

If we assume that $\|b\|_{L^t(\Omega)}$ is small enough, that is we assume that B satisfies (3.18) and moreover

$$\frac{B^2}{\varepsilon_1^p} + \frac{B}{\varepsilon_1^p} < 1, \tag{3.36}$$

then (3.35) and (3.36) give a contradiction. We conclude that $\left|\{x \in \Omega : \ |u(x) - v(x)| > 0\} = 0\right|$, i.e. $u = v$ almost everywhere in Ω.

Step 3. *The case where $\|c\|_{L^t(\Omega)}$ is small enough and $\|b\|_{L^r(\Omega)}$ is large.*

As in Step 2 we begin by observing that the proof of estimate (3.13) in Step 1 is made without any assumption on the smallness of the norms $\|b\|_{L^r(\Omega)}^{p/2}$ or $\|c\|_{L^t(\Omega)}^{p/2}$.

Now we argue by contradiction, i.e. let us assume that

$$\left|\{x \in \Omega : \ |u(x) - v(x)| > 0\}\right| > 0.$$

Moreover assume that $\|c\|_{L^t(\Omega)}$ is small enough, that is

$$\|c\|_{L^t(\Omega)}^{p/2} \le \eta', \tag{3.37}$$

where $\eta' > 0$ is the constant defined in Remark 4. By (2.20) in Remark 4, since $\|c\|_{L^t(\Omega)}^{p/2} \le \eta'$, $\|1 + |u| + |v|\|_{L^{p^*}(\Omega)}^{\tau p/2} \||\nabla u| + |\nabla v|\|_{L^p(\Omega)}^{(2-p)p/2}$ and $\|1 + |\nabla u| + |\nabla v|\|_{L^p(\Omega)}^{\sigma p/2} \||\nabla u| + |\nabla v|\|_{L^p(\Omega)}^{(2-p)p/2}$ are bounded by a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c. Therefore by (3.13) we get

$$\begin{aligned} \|\nabla S_m(u-v)\|_{(L^p(\Omega))^N}^p \le & C_0' \|c\|_{L^t(\Omega)}^{p/2} \|\nabla S_m(u-v)\|_{(L^p(\Omega))^N}^p \\ & + C_0' \|b\|_{L^r(\{|u-v|>m\}\cap Z)}^{p/2} \|\nabla S_m(u-v)\|_{(L^p(\Omega))^N}^p \\ & + C_0' m^{p/2} \|c\|_{L^t(\Omega)}^{p/2} \|\nabla S_m(u-v)\|_{(L^p(\Omega))^N}^{p/2}, \end{aligned} \tag{3.38}$$

where C_0' is a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$ but not on c.

In this Step we assume that (3.14) does not hold and that the following conditions are satisfied

$$C_0' \|b\|_{L^r(\{|u-v|>0\}\cap Z)}^{p/2} > \frac{1}{4} \quad \text{and} \quad \|c\|_{L^t(\Omega)}^{p/2} \le B', \tag{3.39}$$

where B' is a constant small enough, i.e.

$$B' \le \min\left\{\eta', \frac{1}{4C_0'}\right\}, \tag{3.40}$$

and which will be better specified later.

Now let us consider the function $F : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$F(m) = C_0' \|b\|^{p/2}_{L^r(\{|u-v|>m\}\cap Z)}.$$

It is continuous, decreasing, it tends to zero as m goes to infinity and since we assume (3.39), it verifies $F(0) > 1/4$. Therefore there exists a value of m (which will continue to denote by m_1 as in the previous Step, but which can be different of it), i. e. $m = m_1 > 0$ such that

$$F(m_1) = C_0' \|b\|^{p/2}_{L^r(\{|u-v|>m_1\}\cap Z)} = \frac{1}{4}. \tag{3.41}$$

By (3.40) we have $C_0'\|c\|^{p/2}_{L^t(\Omega)} \le C_0' B' \le 1/4$ and then by (3.38) we get

$$\|\nabla S_{m_1}(u-v)\|^p_{(L^p(\Omega))^N} \le 2C_0' {m_1}^{p/2} \|c\|^{p/2}_{L^t(\Omega)} \|\nabla S_{m_1}(u-v)\|^{p/2}_{(L^p(\Omega))^N},$$

or, equivalently,

$$\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N} \le (2C_0')^{2/p} m_1 \|c\|_{L^t(\Omega)}. \tag{3.42}$$

We now derive a "log-type estimate" on $u - v$. Denote by $\phi : (-M, M) \mapsto \mathbb{R}$ the function defined by

$$\phi(w) = \int_0^w \frac{ds}{(M - |s|)^2}, \qquad \forall w \in (-M, M), \tag{3.43}$$

where M is a positive constant such that $M > m_1$ and which will be specified later.

Observe that, since $(u - v)$ belongs to $W_0^{1,p}(\Omega)$ and since $\phi \circ T_{m_1}$ is a Lipschitz function such that $\phi \circ T_{m_1}(0) = 0$, then the function $\phi(T_{m_1}(u - v))$ belongs to $W_0^{1,p}(\Omega)$. Moreover, by the definition of ϕ, we have

$$\big|\phi(T_{m_1}(u-v))\big| = \int_0^{|T_{m_1}(u-v)|} \frac{ds}{(M-|s|)^2} = \frac{|T_{m_1}(u-v)|}{M(M - |T_{m_1}(u-v)|)}, \tag{3.44}$$

and therefore

$$\big|\phi(T_{m_1}(u-v))\big| \le \frac{m_1}{M(M-m_1)}. \tag{3.45}$$

Let us choose $\phi(T_{m_1}(u - v))$ as a test function in the equality (2.12) satisfied by u and in the equality (2.12) satisfied by v. By subtracting the two results, we get

$$\begin{aligned}\int_{\{|u-v|<m_1\}} &(a(x,\nabla u) - a(x,\nabla v)) \cdot \nabla(u-v)\phi'(T_{m_1}(u-v)) \\ &+ \int_{\{|u-v|<m_1\}} (\Phi(x,u) - \Phi(x,v)) \cdot \nabla(u-v)\phi'(T_{m_1}(u-v)) \\ &\qquad + \int_\Omega (H(x,\nabla u) - H(x,\nabla v))\phi(T_{m_1}(u-v)) = 0.\end{aligned} \tag{3.46}$$

By the monotonicity assumption (2.4) on a, the assumptions of locally Lipschitz continuity (2.9) on Φ and (2.10) on H, and since $\phi'(w) = \frac{1}{(M-|w|)^2}$ for any $w \in$

$(-M, M)$, we get

$$\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \frac{1}{(|\nabla u|+|\nabla v|)^{2-p}} \\ \leq \int_{\{|u-v|<m_1\}} c(x)(1+|u|+|v|)^\tau |u-v| \frac{|\nabla(u-v)|}{(M-|u-v|)^2} \\ + \int_\Omega b(x)(1+|u|+|v|)^\sigma |\nabla u - \nabla v||\phi(T_{m_1}(u-v))|.$$

As in the previous steps, we define the set Z. Since by (3.4) $\nabla u - \nabla v = 0$ almost everywhere on Z^c, we have

$$\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \frac{1}{(|\nabla u|+|\nabla v|)^{2-p}} \leq J_1' + J_2', \tag{3.47}$$

where

$$J_1' = \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau |u-v| \frac{|\nabla(u-v)|}{(M-|u-v|)^2},$$

$$J_2' = \int_Z b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla u - \nabla v||\phi(T_{m_1}(u-v))|.$$

Now we will estimate J_1' and J_2'. We first deal with J_1'. Since $M - |T_{m_1}(u-v)| \geq M - m_1$, we have

$$J_1' \leq \frac{m_1}{M-m_1} \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau \frac{|\nabla(u-v)|}{[M-|u-v|]}. \tag{3.48}$$

Since $p < N$, assumption (2.17) on τ implies that

$$\frac{1}{t} + \frac{\tau}{p^*} + \frac{1}{p} < 1.$$

Therefore we can apply Hölder's inequality in (3.48) and we get

$$J_1' \leq \frac{m_1}{M-m_1} |\Omega|^{1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{p}} \|c\|_{L^t(\Omega)} \|1+|u|+|v|\|^\tau_{L^{p^*}(\Omega)} \\ \times \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{[M-|u-v|]^p} \right)^{1/p}. \tag{3.49}$$

Now we evaluate J_2'. By (3.44), since $M > m_1$, then $\frac{|T_{m_1}(u-v)|}{M} \leq 1$. Therefore we get

$$J_2' \leq \int_{\{|u-v|<m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma \frac{|\nabla(u-v)|}{[M-|u-v|]} \\ + \frac{m_1}{M(M-m_1)} \int_{\{|u-v|>m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla S_{m_1}(u-v)|. \tag{3.50}$$

Moreover, since $p < N$, assumption (2.18) on σ implies that

$$\frac{1}{r} + \frac{\sigma}{p} + \frac{1}{p} < 1.$$

Therefore, using Hölder's inequality in (3.50) we obtain

$$
\begin{aligned}
J_2' \leq{}& |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\Omega)} \|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \\
&\times \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla T_{m_1}(u-v)|^p}{[M-|T_{m_1}(u-v)|]^p} \right)^{1/p} \\
&+ \frac{m_1}{M(M-m_1)} |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\{|u-v|>m_1\}\cap Z)} \\
&\times \|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}.
\end{aligned}
$$

Combining (3.47), (3.49) and (3.50) yields that

$$
\begin{aligned}
\beta \int_{\{|u-v|<m_1\}} & \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \frac{1}{(|\nabla u|+|\nabla v|)^{2-p}} \\
&\leq \frac{m_1}{M-m_1} |\Omega|^{1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{p}} \|c\|_{L^t(\Omega)} \|1+|u|+|v|\|^{\tau}_{L^{p^*}(\Omega)} \\
&\quad \times \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M-|u-v|)^p} \right)^{1/p} \\
&\quad + |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\Omega)} \|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \\
&\quad \times \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M-|u-v|)^p} \right)^{1/p} \\
&\quad + \frac{m_1}{M(M-m_1)} |\Omega|^{1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p}} \|b\|_{L^r(\{|u-v|>m_1\}\cap Z)} \\
&\quad \times \|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}.
\end{aligned} \tag{3.51}
$$

On the other hand by Hölder's inequality, since $p \leq 2$, we get

$$
\begin{aligned}
\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M-|u-v|)^p} \leq{}& \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \frac{1}{(|\nabla u|+|\nabla v|)^{2-p}} \right)^{p/2} \\
&\times \left(\int_{\Omega} (|\nabla u|+|\nabla v|)^p \right)^{(2-p)/2}.
\end{aligned} \tag{3.52}
$$

Gathering (3.51) and (3.52), we obtain

$$\begin{aligned}\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M-|u-v|)^p} &\le \||\nabla u|+|\nabla v|\|_{L^p(\Omega)}^{(2-p)p/2}\frac{1}{\beta^{p/2}}\\ &\times\Big\{\frac{(m_1)^{p/2}}{(M-m_1)^{p/2}}|\Omega|^{(1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{p})\frac{p}{2}}\|c\|_{L^t(\Omega)}^{p/2}\|1+|u|+|v|\|_{L^{p^*}(\Omega)}^{\tau p/2}\\ &\times\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^p}{(M-|u-v|)^p}\right)^{1/2}\\ &+|\Omega|^{(1-\frac{1}{r}-\frac{\sigma}{p}-\frac{1}{p})\frac{p}{2}}\|b\|_{L^r(\Omega)}^{p/2}\|1+|\nabla u|+|\nabla v|\|_{L^p(\Omega)}^{\sigma p/2}\\ &\times\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^p}{(M-|u-v|)^p}\right)^{1/2}\\ &+\frac{(m_1)^{p/2}}{M^{p/2}(M-m_1)^{p/2}}|\Omega|^{(1-\frac{1}{t}-\frac{\sigma}{p}-\frac{1}{p})\frac{p}{2}}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}^{p/2}\\ &\times\|1+|\nabla u|+|\nabla v|\|_{L^p(\Omega)}^{\sigma p/2}\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}^{p/2}\Big\}.\end{aligned}\tag{3.53}$$

Now we observe that, by (2.20) in Remark 4 and (3.39), since we chose in (3.40) B' in such a way that $B'\le\eta'$, the terms $\||\nabla u|+|\nabla v|\|_{L^p(\Omega)}^{(2-p)p/2}\|1+|u|+|v|\|_{L^{p^*}(\Omega)}^{\tau p/2}$ and $\||\nabla u|+|\nabla v|\|_{L^p(\Omega)}^{(2-p)p/2}\|1+|\nabla u|+|\nabla v|\|_{L^p(\Omega)}^{\sigma p/2}$ are bounded by a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c. Therefore by (3.53), we get

$$\begin{aligned}&\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M-|u-v|)^p}\\ &\le C_1'\frac{(m_1)^{p/2}}{(M-m_1)^{p/2}}\|c\|_{L^t(\Omega)}^{p/2}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^p}{(M-|u-v|)^p}\right)^{1/2}\\ &\quad+C_1'\|b\|_{L^r(\Omega)}^{p/2}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^p}{[M-|u-v|]^p}\right)^{1/2}\\ &\quad+C_1'\frac{(m_1)^{p/2}}{M^{p/2}(M-m_1)^{p/2}}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}^{p/2}\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}^{p/2},\end{aligned}\tag{3.54}$$

where C_1' is a positive constant which depends on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^N(\Omega)}$, but not on c.

On the other hand using the estimate (3.42) of $\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}$, since m_1 verifies (3.41), i.e. $C_0'\|b\|_{L^r(\{|u-v|>m\}\cap Z)}^{p/2}=\frac{1}{4}$, we have

$$\begin{aligned}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}^{p/2}\|\nabla S_{m_1}(u-v)\|_{(L^p(\Omega))^N}^{p/2} &\le 2\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}^{p/2}C_0'm_1^{p/2}\|c\|_{L^t(\Omega)}^{p/2}\\ &\le\frac{1}{2}m_1^{p/2}\|c\|_{L^t(\Omega)}^{p/2}.\end{aligned}$$

Thanks to the above inequality together with Young's inequality, (3.54) leads to

$$\begin{aligned}\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{[M-|u-v|]^p} \le C_2'\left(\frac{m_1}{M-m_1}\right)^p \|c\|^p_{L^t(\Omega)} + C_2'\|b\|^p_{L^r(\Omega)}\\ + C_2'\left(\frac{m_1^p}{M^{p/2}(M-m_1)^{p/2}}\right)\|c\|^{p/2}_{L^t(\Omega)},\end{aligned} \tag{3.55}$$

where C_2' is a positive constant which depends on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c.

Now let us observe that the equiintegrability of b implies the existence of a positive constant δ_0, depending on b and on C_0' (but not on c), such that

$$|\{|u-v|>m_1\}\cap Z| \ge \delta_0.$$

Therefore we have

$$\delta_0 \le \delta = \big|(Z\cap\{|u-v|>m_1\})\big| \le \big|\{|u-v|>m_1\}\big|.$$

By using Poincaré's inequality, we get

$$\begin{aligned}&\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^p}{(M-|u-v|)^p} = \int_\Omega \frac{|\nabla T_{m_1}(u-v)|^p}{(M-|T_{m_1}(u-v)|)^p}\\ &= \int_\Omega \Big|\nabla \ln\big(1-\frac{|T_{m_1}(u-v)|}{M}\big)\Big|^p \ge C(|\Omega|,p)\int_\Omega \Big|\ln\big(1-\frac{|T_{m_1}(u-v)|}{M}\big)\Big|^p\\ &\ge C(|\Omega|,p)\int_{\{|u-v|>m_1\}} \Big|\ln\big(1-\frac{|T_{m_1}(u-v)|}{M}\big)\Big|^p \ge C(|\Omega|,p)\delta_0\Big|\ln\big(1-\frac{m_1}{M}\big)\Big|^p.\end{aligned} \tag{3.56}$$

Combining (3.56) and (3.55) we obtain

$$\begin{aligned}C(|\Omega|,p)\delta_0\Big|\ln\Big(1-\frac{m_1}{M}\Big)\Big|^p \le C_2'\frac{(m_1)^p}{(M-m_1)^p}\|c\|^p_{L^t(\Omega)} + C_2'\|b\|^p_{L^r(\Omega)}\\ + C_2'\frac{m_1^p}{M^{p/2}(M-m_1)^{p/2}}\|c\|^{p/2}_{L^t(\Omega)}.\end{aligned} \tag{3.57}$$

Now let us observe that, since M is a positive constant to be specified such that $M>m_1$, we can denote

$$M = m_1(1+\varepsilon),$$

where $\varepsilon>0$ will be chosen later. Therefore by (3.57), we get

$$C(|\Omega|,p)\delta_0\Big|\ln\Big(1+\frac{1}{\varepsilon}\Big)\Big|^p \le \frac{C_2'}{\varepsilon^p}\|c\|^p_{L^t(\Omega)} + C_2'\|b\|^p_{L^r(\Omega)} + \frac{C_2'}{\varepsilon^{p/2}}\|c\|^{p/2}_{L^t(\Omega)}$$

and since by (3.39) $\|c\|^{p/2}_{L^t(\Omega)} \le B'$, we have

$$\delta_0\Big[\ln\Big(1+\frac{1}{\varepsilon}\Big)\Big]^p \le C_3'\frac{B'^2}{\varepsilon^p} + C_3' + C_3'\frac{B'}{\varepsilon^{p/2}}, \tag{3.58}$$

where C_3' is a positive constant which depends on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c.

Now we choose $\varepsilon=\varepsilon_1>0$ in such a way that

$$\delta_0\Big[\ln\Big(1+\frac{1}{\varepsilon_1}\Big)\Big]^p \ge 2C_3'.$$

It is worth noting that ε_1 does not depend on c, u and v. Then (3.58) implies

$$2C_3' \leq C_3' \frac{B'^2}{{\varepsilon_1}^p} + C_3' + C_3' \frac{B'}{{\varepsilon_1}^{p/2}},$$

or, equivalently,

$$1 \leq \frac{B'^2}{{\varepsilon_1}^p} + \frac{B'}{{\varepsilon_1}^{p/2}}. \tag{3.59}$$

If we assume that $\|c\|_{L^t(\Omega)}$ is small enough, that is we assume that B' satisfies (3.40) and moreover

$$\frac{B'^2}{{\varepsilon_1}^p} + \frac{B'}{{\varepsilon_1}^{p/2}} < 1, \tag{3.60}$$

then (3.59) and (3.60) give a contradiction. Then we conclude that $\big|\{x \in \Omega : |u(x) - v(x)| > 0\}\big| = 0$, i.e. $u = v$ almost everywhere in Ω.

The proof of Theorem 2.1 is complete.

3.2. **Proof of Theorem 2.2.** The strategy of the proof of Theorem 2.2 is the same as that of Theorem 2.1, i.e. it is performed in three steps which correspond to the case where $\|c\|_{L^t(\Omega)}$ and $\|b\|_{L^r(\Omega)}$ are small enough, the case where $\|c\|_{L^t(\Omega)}$ is small enough and $\|b\|_{L^r(\Omega)}$ is large and the case where $\|c\|_{L^t(\Omega)}$ is large and $\|b\|_{L^r(\Omega)}$ is small enough. There are technical differences due to the assumption (2.4) on the "strong monotonicity" of the operator a. We explicitly remark that the proofs of Theorems 2.1 and 2.2 coincide when $p = 2$.

Step 1. *The case where $\|c\|_{L^t(\Omega)}$ and $\|b\|_{L^r(\Omega)}$ are small enough.*

For $m > 0$ consider the function $S_m(s)$ defined in (3.1), with $m > 0$ which will be specified later. As at the beginning of the Step 1 in the proof of Theorem 2.1, we arrive to equality (3.2), which is

$$\int_\Omega (a(x,\nabla u) - a(x, \nabla v)) \cdot \nabla S_m(u-v) + \int_\Omega (\Phi(x,u) - \Phi(x,v)) \cdot \nabla S_m(u-v) \\ + \int_\Omega (H(x,\nabla u) - H(x,\nabla v)) S_m(u-v) = 0.$$

By the monotonicity assumption (2.4) on a and the assumptions of locally Lipschitz continuity (2.9) on Φ and (2.10) on H, we get

$$\beta \int_{\{|u-v|>m\}} |\nabla(u-v)|^2 (1 + |\nabla u| + |\nabla v|)^{p-2} \\ \leq \int_{\{|u-v|>m\}} c(x)(1+|u|+|v|)^\tau |u-v| |\nabla S_m(u-v)| \\ + \int_{\{|u-v|>m\}} b(x)(1+|\nabla u| + |\nabla v|)^\sigma |\nabla(u-v)| |S_m(u-v)|. \tag{3.61}$$

As in the proof of Theorem 2.1, we define the set Z. As $|\Omega|$ is finite, the set of the constants k such that $|\{x \in \Omega : \ |(u-v)(x)| = k\}| > 0$ is at most countable. Let Z^c be the (countable) union of all those sets. Its complementary $Z = \Omega \setminus Z^c$ is therefore the union of the sets such that $|\{x \in \Omega : \ |(u-v)(x)| = k\}| = 0$. Since for every k,

$$\nabla(u-v) = 0 \qquad \text{a.e. on } \{x \in \Omega, \ |(u-v)(x)| = k\},$$

and since Z^c is at most a countable union, we obtain that

$$\nabla u - \nabla v = 0 \quad \text{a.e. on } Z^c. \tag{3.62}$$

We deduce that

$$\beta\int_{\{|u-v|>m\}}|\nabla(u-v)|^2(1+|\nabla u|+|\nabla v|)^{p-2}\le I_1+I_2 \tag{3.63}$$

with

$$I_1=\int_{\{|u-v|>m\}\cap Z}c(x)(1+|u|+|v|)^{\tau}|u-v||\nabla S_m(u-v)|, \tag{3.64}$$

$$I_2=\int_{\{|u-v|>m\}\cap Z}b(x)(1+|\nabla u|+|\nabla v|)^{\sigma}|\nabla(u-v)||S_m(u-v)|. \tag{3.65}$$

Now we evaluate I_1 and I_2. As far as I_1 is concerned, we have

$$\begin{aligned}I_1&=\int_{\{|u-v|>m\}\cap Z}c(x)(1+|u|+|v|)^{\tau}|u-v||\nabla S_m(u-v)|\\&\le\int_{\{|u-v|>m\}\cap Z}c(x)(1+|u|+|v|)^{\tau}|S_m(u-v)||\nabla S_m(u-v)|\\&\quad+m\int_{\{|u-v|>m\}\cap Z}c(x)(1+|u|+|v|)^{\tau}|\nabla S_m(u-v)|.\end{aligned} \tag{3.66}$$

We now claim that assumption (2.24) on τ leads to

$$\begin{aligned}I_1&\le S(N,p,t,\Omega,\tau)\|c\|_{L^t(\{|u-v|>m\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\tau}_{L^p(\Omega)}\\&\quad\times\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}\Big(\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}+m\Big)\end{aligned} \tag{3.67}$$

where $S(N,p,t,\Omega,\tau)$ is a constant which depends on N, p, t, Ω and τ.

Indeed in the case where $p<N$, since $2<N$, then assumption (2.24) on τ is equivalent to

$$\frac{1}{t}+\frac{\tau}{p^*}+\frac{1}{2^*}+\frac{1}{2}\le 1.$$

Therefore Hölder's inequality applied in (3.66) together with Sobolev's embedding theorem give

$$\begin{aligned}I_1&\le\|c\|_{L^t(\{|u-v|>m\}\cap Z)}\|1+|u|+|v|\|^{\tau}_{L^{p^*}(\Omega)}S_{N,2}\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}\\&\quad\times\Big(\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}+m|\Omega|^{1-\frac{1}{t}-\frac{\tau}{p^*}-\frac{1}{2}}\Big)\end{aligned} \tag{3.68}$$

where $S_{N,2}$ is the best constant in the embedding of $W_0^{1,2}(\Omega)$ into $L^{2^*}(\Omega)$. Using the Sobolev embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ we deduce that inequality (3.67) holds true.

If $N=p$, due to (2.24) the value of τ is any positive real number when $N<t$ and it is equal to zero when $N=t$. It follows that in both cases there exists $q\ge 1$ such that

$$\frac{1}{t}+\frac{\tau}{q}+\frac{1}{2^*}+\frac{1}{2}\le 1.$$

Therefore from (3.66) we obtain

$$\begin{aligned}I_1&\le\|c\|_{L^t(\{|u-v|>m\}\cap Z)}\|1+|u|+|v|\|^{\tau}_{L^{q}(\Omega)}S_{N,2}\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}\\&\quad\times\Big(\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}+m|\Omega|^{1-\frac{1}{t}-\frac{\tau}{q}-\frac{1}{2}}\Big)\end{aligned} \tag{3.69}$$

and the Sobolev embedding theorem of $W_0^{1,p}(\Omega)$ (here $p=N$) into $L^q(\Omega)$ leads to (3.67).

Now we evaluate I_2. Since $2 < N$, assumption (2.25) on σ is equivalent to

$$\frac{1}{r} + \frac{\sigma}{p} + \frac{1}{2} + \frac{1}{2^*} \leq 1.$$

Therefore, thanks to Hölder's inequality and Sobolev's embedding theorem, (3.65) gives

$$I_2 \leq \|b\|_{L^r(\{u-v|>m\}\cap Z)} \|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^p(\Omega)} S_{N,2} \|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N}. \quad (3.70)$$

Since $(1 + |\nabla u| + |\nabla v|)^{p-2} \geq 1$ a. e. in Ω, combining (3.63), (3.67) and (3.70) leads to

$$\begin{aligned}\beta \int_\Omega |\nabla S_m(u-v)|^2 \leq{}& S(N,p,t,\Omega,\tau)\|c\|_{L^t(\{|u-v|>m\}\cap Z)} \|1 + |\nabla u| + |\nabla v|\|^{\tau}_{L^p(\Omega)} \\ &\times \|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N} \\ &+ S_{N,2}\|b\|_{L^r(\{u-v|>m\}\cap Z)} \|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^p(\Omega)} \|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N} \\ &+ S(N,p,t,\Omega,\tau) m \|c\|_{L^t(\{|u-v|>m\}\cap Z)} \|1 + |\nabla u| + |\nabla v|\|^{\tau}_{L^p(\Omega)} \\ &\times \|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}.\end{aligned} \quad (3.71)$$

Now we argue by contradiction. Let us assume that

$$\big|\{x \in \Omega : |u(x) - v(x)| > 0\}\big| > 0$$

and that

$$\|c\|_{L^t(\{|u-v|>0\}\cap Z)} \quad \text{and} \quad \|b\|_{L^r(\{u-v|>0\}\cap Z)} \text{ are small enough,}$$

i.e.

$$\begin{cases} S(N,p,t,\Omega,\tau)\|c\|_{L^t(\{|u-v|>0\}\cap Z)} \|1 + |\nabla u| + |\nabla v|\|^{\tau}_{L^p(\Omega)} \leq \dfrac{1}{4}, \\ S_{N,2}\|b\|_{L^r(\{u-v|>0\}\cap Z)} \|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^p(\Omega)} \leq \dfrac{1}{4}. \end{cases} \quad (3.72)$$

Then we can choose $m = 0$ in (3.71) and we get

$$\|\nabla(u-v)\|^2_{(L^2(\Omega))^N} \leq 0,$$

which is a contradiction. Therefore we conclude that $\big|\{x \in \Omega : |u(x) - v(x)| > 0\}\big| = 0$, i.e. $u = v$ a. e. in Ω.

Step 2. *The case where $\|c\|_{L^t(\Omega)}$ is large and $\|b\|_{L^r(\Omega)}$ is small enough.*

We begin by observing that the proof of estimate (3.71) in the previous Step is made without any assumption on the smallness of the norms $\|b\|^{p/2}_{L^r(\Omega)}$ or $\|c\|^{p/2}_{L^t(\Omega)}$.

Let us assume that $\|b\|_{L^r(\Omega)}$ is small enough, that is

$$\|b\|^{p/2}_{L^r(\Omega)} \leq \eta, \quad (3.73)$$

where $\eta > 0$ is the constant defined in Remark 4. By (2.19) in Remark 4, the terms $\|1 + |u| + |v|\|^{\tau p/2}_{L^{p^*}(\Omega)} \||\nabla u| + |\nabla v|\|^{(2-p)2/p}_{L^p(\Omega)}$ and $\|1 + |\nabla u| + |\nabla v|\|^{\sigma p/2}_{L^p(\Omega)} \||\nabla u| + |\nabla v|\|^{(2-p)2/p}_{L^p(\Omega)}$ are bounded by a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

Hence (3.73) and (3.71) give that

$$\begin{aligned}\int_\Omega |\nabla S_m(u-v)|^2 \le C_0 \|c\|_{L^t(\{|u-v|>m\}\cap Z)} &\|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N} \\ &+ C_0\|b\|_{L^r(\{u-v|>m\}\cap Z)}\|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N} \\ &+ mC_0\|c\|_{L^t(\{|u-v|>m\}\cap Z)}\|\nabla S_m(u-v)\|_{(L^2(\Omega))^N},\end{aligned} \tag{3.74}$$

where C_0 is a positive constant which depends only on β, N, $|\Omega|$, p, α, τ, t, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

In this Step we assume that (3.72) does not hold and that the following conditions are satisfied

$$C_0\|c\|_{L^t(\{|u-v|>0\}\cap Z)} > \frac{1}{4} \quad \text{and} \quad \|b\|_{L^r(\Omega)} \le B, \tag{3.75}$$

where B is a constant small enough, i.e.

$$B \le \min\left\{\eta, \frac{1}{4C_0}\right\}, \tag{3.76}$$

and which will be better specified later.

Let us consider the function $G:\mathbb{R}\mapsto\mathbb{R}$ defined by

$$G(m) = C_0\|c\|_{L^t(\{|u-v|>m\}\cap Z)}$$

which is continuous, decreasing and tends to zero as m goes to infinity and, since we assume (3.75), it verifies $G(0) > 1/4$. Therefore there exists $m = m_1 > 0$ such that

$$G(m_1) = C_0\|c\|_{L^t(\{|u-v|>m_1\}\cap Z)} = \frac{1}{4}.$$

By (3.76) we have $C_0\|b\|^{p/2}_{L^r(\Omega)} \le C_0 B \le 1/4$ and then therefore by (3.74) we have

$$\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N} \le \frac{m_1}{2}. \tag{3.77}$$

Now we derive a technical "log-type estimate" on $u-v$. Denote by $\varphi:\mathbb{R}\mapsto\mathbb{R}$ the function defined by (3.20). Let us choose $\varphi(T_{m_1}(u-v))$ as test function in the equality (2.12) satisfied by u and in the equality (2.12) satisfied by v. By subtracting the two results, we get (3.22), that is

$$\begin{aligned}&\int_{\{|u-v|<m_1\}} (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla(u-v)\varphi'(T_{m_1}(u-v)) \\ &\qquad + \int_{\{|u-v|<m_1\}} (\Phi(x,u)-\Phi(x,v))\cdot\nabla(u-v)\varphi'(T_{m_1}(u-v)) \\ &\qquad\qquad + \int_\Omega (H(x,\nabla u)-H(x,\nabla v))\varphi(T_{m_1}(u-v)) = 0.\end{aligned}$$

By the monotonicity assumption (2.4) on a, the assumptions of locally Lipschitz continuity (2.9) on Φ and (2.10) on H, and since $\varphi'(w) = \frac{1}{(M+|w|)^2}$, we get

$$\begin{aligned}\beta\int_{\{|u-v|<m_1\}} &\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}(1+|\nabla u|+|\nabla v|)^{p-2} \\ &\le \int_{\{|u-v|<m_1\}} c(x)(1+|u|+|v|)^\tau|u-v|\frac{|\nabla(u-v)|}{(M+|u-v|)^2} \\ &\quad + \int_\Omega b(x)(1+|\nabla u|+|\nabla v|)^\sigma|\nabla u-\nabla v||\varphi(T_{m_1}(u-v))|.\end{aligned}$$

As in Step 1 we define the set Z. Since by (3.4), $\nabla u - \nabla v = 0$ a. e. on Z^c, we deduce

$$\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}(1+|\nabla u|+|\nabla v|)^{p-2} \le J_1 + J_2, \tag{3.78}$$

with

$$J_1 = \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau |u-v| \frac{|\nabla(u-v)|}{(M+|u-v|)^2},$$
$$J_2 = \int_Z b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla u - \nabla v| \varphi(T_{m_1}(u-v)).$$

Now we estimate J_1 and J_2. As far as J_1 is concerned, using property (3.62) and the fact that $\frac{|T_{m_1}(u-v)|}{M+|T_{m_1}(u-v)|} \le 1$ we get

$$J_1 \le \int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau \frac{|\nabla(u-v)|}{M+|u-v|}. \tag{3.79}$$

With arguments already used to derive inequality (3.67) which consist to distinguish the cases $p<N$ and $p=N$ and to apply the Sobolev embedding theorem we obtain that

$$\begin{aligned} J_1 \le & S(N,p,t,\Omega,\tau)\|c\|_{L^t(\Omega)} \|1+|\nabla u|+|\nabla v|\|^\tau_{L^p(\Omega)} \\ & \times \left(\int_{\{|u-v|<m_1\}\cap Z} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \right)^{1/2}, \end{aligned} \tag{3.80}$$

where $S(N,p,t,\Omega,\tau)$ is a constant depending on N, p, t, Ω and τ.

We now deals with J_2. By the property (3.21) of φ, since $M+m_1 > M+|u-v|$ a. e. on the set $\{|u-v|<m_1\}$, we have

$$\begin{aligned} J_2 \le & \frac{m_1}{M(M+m_1)} \int_{\{|u-v|<m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla(u-v)| \\ & + \frac{m_1}{M(M+m_1)} \int_{\{|u-v|>m_1\}\cap Z} b(x)|\nabla S_{m_1}(u-v)| \\ \le & \frac{m_1}{M} \int_{\{|u-v|<m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma \frac{|\nabla(u-v)|}{M+|u-v|} \\ & + \frac{m_1}{M(M+m_1)} \int_{\{|u-v|>m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla S_{m_1}(u-v)|. \end{aligned} \tag{3.81}$$

On the other hand the assumption (2.25) on σ implies that

$$\frac{1}{r} + \frac{\sigma}{p} + \frac{1}{2} < 1.$$

Therefore we apply Hölder's inequality in (3.81) and we obtain

$$\begin{aligned} J_2 \le & \frac{m_1}{M}|\Omega|^{\frac12 - \frac1r - \frac{\sigma}{p}} \|b\|_{L^r(\{|u-v|<m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)} \\ & \times \left(\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \right)^{1/2} \\ & + \frac{m_1}{M(M+m_1)}|\Omega|^{\frac12 - \frac1r - \frac{\sigma}{p}} \|b\|_{L^r(\{|u-v|>m_1\}\cap Z)} \|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)} \\ & \times \|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N}. \end{aligned} \tag{3.82}$$

Combining (3.78), (3.80) and (3.82) gives

$$
\begin{aligned}
&\beta \int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}(1+|\nabla u|+|\nabla v|)^{p-2} \\
&\le S(N,p,t,\Omega,\tau)\|c\|_{L^t(\Omega)}\|1+|\nabla u|+|\nabla v|\|^{\tau}_{L^p(\Omega)} \\
&\quad\times\left(\int_{\{|u-v|<m_1\}\cap Z}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}\right)^{1/2} \\
&\quad+\frac{m_1}{M}|\Omega|^{\frac12-\frac1r-\frac{\sigma}{p}}\|b\|_{L^r(\{|u-v|<m_1\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \\
&\quad\times\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}\right)^{1/2} \\
&\quad+\frac{m_1}{M(M+m_1)}|\Omega|^{\frac12-\frac1r-\frac{\sigma}{p}}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)} \\
&\quad\times\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N}.
\end{aligned}
\tag{3.83}
$$

Now we observe that, by (2.19) in Remark 4 and (3.75), since we chose in (3.76) B in such a way that $B\le\eta$, the terms $\|1+|\nabla u|+|\nabla v|\|^{\tau}_{L^p(\Omega)}$ and $\|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^p(\Omega)}$ are bounded by a constant which depends only on N, $|\Omega|$, p, τ, σ, α , $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b. Therefore by (3.83), since $1+|\nabla u|+|\nabla v|\ge 1$ a. e. in Ω, we deduce that

$$
\begin{aligned}
\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} &\le C_1\|c\|_{L^t(\Omega)}\left(\int_{\{|u-v|<m_1\}\cap Z}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}\right)^{1/2} \\
&\quad+C_1\frac{m_1}{M}\|b\|_{L^r(\{|u-v|<m_1\}\cap Z)}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}\right)^{1/2} \\
&\quad+C_1\frac{m_1}{M(M+m_1)}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}\|\nabla S_{m_1}(u-v)\|_{L^2(\Omega)},
\end{aligned}
$$

where C_1 is a constant which depends only on β, N, $|\Omega|$, p, τ, t, σ, α, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b. Moreover by Young's inequality and by estimate (3.77) on $\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N}$ we get

$$
\begin{aligned}
\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} &\le (C_1)^2\|c\|^2_{L^t(\Omega)}+\frac14\int_{\{|u-v|<m_1\}\cap Z}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \\
&\quad+(C_1)^2\frac{m_1^2}{M^2}\|b\|^2_{L^r(\{|u-v|<m_1\}\cap Z)}+\frac14\int_{\{|u-v|<m_1\}\cap Z}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2} \\
&\quad+\frac{C_1}{2}\frac{m_1^2}{M(M+m_1)}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}.
\end{aligned}
$$

This implies

$$
\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M+|u-v|)^2}\le C_2\|c\|^2_{L^t(\Omega)}+C_2\frac{m_1^2}{M^2}\|b\|^2_{L^r(\Omega)}+C_2\frac{m_1^2}{M^2}\|b\|_{L^r(\Omega)},
\tag{3.84}
$$

where C_2 is a constant which depends only on β, N, $|\Omega|$, p, τ, t, α, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|c\|_{L^t(\Omega)}$, but not on b.

Observe that inequality (3.84) coincides with inequality (3.31) in the proof of Theorem 2.1 (in the case $\|c\|_{L^t(\Omega)}$ large and $\|b\|_{L^r(\Omega)}$ small enough) when p is replaced by 2. With arguments already used it follows that we obtain the same conclusion of Step 2 in the proof of Theorem 2.1 : there exists $B > 0$ such that if $\|b\|_{L^r(\Omega)}$ is small enough, i.e. $\|b\|_{L^r(\Omega)} \leq B$, then $u = v$ almost everywhere in Ω.

Step 3. *The case where $\|c\|_{L^t(\Omega)}$ is small enough and $\|b\|_{L^r(\Omega)}$ is large.*

As in Step 2 we begin by observing that the proof of estimate (3.71) in Step 1 is made without any assumption on the smallness of the norms $\|b\|^{p/2}_{L^r(\Omega)}$ or $\|c\|^{p/2}_{L^t(\Omega)}$.

Let us assume that $\|c\|_{L^t(\Omega)}$ is small enough, that is

$$\|c\|_{L^t(\Omega)} \leq \eta' \tag{3.85}$$

where $\eta' > 0$ is the constant defined in Remark 4.

By (2.19) in Remark 4, since $\|c\|^{p/2}_{L^t(\Omega)} \leq \eta'$, $\||\nabla u| + |\nabla v|\|^{\tau}_{L^p(\Omega)}$ and $\||\nabla u| + |\nabla v|\|^{\sigma}_{L^p(\Omega)}$ are bounded by a positive constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c. Hence (3.85) and (3.71) give

$$\begin{aligned}\int_\Omega |\nabla S_m(u-v)|^2 \leq C_0' \|c\|_{L^t(\{|u-v|>m\}\cap Z)} \|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N} \\ + C_0' \|b\|_{L^r(\{u-v|>m\}\cap Z)} \|\nabla S_m(u-v)\|^2_{(L^2(\Omega))^N} \\ + mC_0' \|c\|_{L^t(\{|u-v|>m\}\cap Z)} \|\nabla S_m(u-v)\|_{(L^2(\Omega))^N}\end{aligned} \tag{3.86}$$

where C_0' is a positive constant which depends only on β, N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c.

In this Step we assume that (3.72) does not hold and that the following conditions are satisfied

$$C_0' \|b\|_{L^r(\{|u-v|>0\}\cap Z)} > \frac{1}{4} \quad \text{and} \quad \|c\|_{L^r(\Omega)} \leq B', \tag{3.87}$$

where B' is a constant small enough, i.e.

$$B' \leq \min\left\{\eta', \frac{1}{4C_0'}\right\}, \tag{3.88}$$

and which will be better specified later.

Now let us consider the function $F : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$F(m) = C_0' \|b\|^{p/2}_{L^r(\{|u-v|>m\}\cap Z)}.$$

It is continuous, decreasing, it tends to zero as m goes to infinity and since we assume (3.87), it verifies $F(0) > 1/4$. Therefore there exists a value of m (which will continue to denote by m_1 as in the previous Step, but which can be different of it), i. e. $m = m_1 > 0$ such that

$$F(m_1) = C_0' \|b\|^{p/2}_{L^r(\{|u-v|>m\}\cap Z)} = \frac{1}{4}.$$

By (3.88) we have $C_0' \|b\|^{p/2}_{L^t(\Omega)} \leq C_0' B' \leq 1/4$ and therefore by (3.86) we get

$$\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N} \leq 2C_0' m_1 \|c\|_{L^t(\Omega)}. \tag{3.89}$$

We now derive a "log-type estimate" on $u - v$. As in the Step 3 of the proof of Theorem 2.1, we consider the function ϕ defined in (3.43) and we choose $\phi(T_{m_1}(u -$

$v)$) as a test function in the equality (2.12) satisfied by u and in the equality (2.12) satisfied by v. By subtracting the two results, we get equation (3.46) which is

$$\int_{\{|u-v|<m_1\}} (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla(u-v)\phi'(T_{m_1}(u-v)) + \int_{\{|u-v|<m_1\}} (\Phi(x,u)-\Phi(x,v))\cdot\nabla(u-v)\phi'(T_{m_1}(u-v)) + \int_\Omega (H(x,\nabla u)-H(x,\nabla v))\phi(T_{m_1}(u-v))=0.$$

By the strong monotonicity assumption (2.4) on a, the assumptions of locally Lipschitz continuity (2.9) on Φ and (2.10) on H and since $\phi'(w)=\frac{1}{(M-|w|)^2}$ for any $w\in(-M,M)$, we get

$$\beta\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}(1+|\nabla u|+|\nabla v|)^{p-2} \le \int_{\{|u-v|<m_1\}} c(x)(1+|u|+|v|)^\tau|u-v|\frac{|\nabla(u-v)|}{(M-|u-v|)^2} + \int_\Omega b(x)(1+|u|+|v|)^\sigma|\nabla u-\nabla v||\phi(T_{m_1}(u-v))|.$$

As in the previous steps, we define the set Z. Since by (3.4) $\nabla u-\nabla v=0$ almost everywhere on Z^c, we have

$$\beta\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}(1+|\nabla u|+|\nabla v|)^{p-2}\le J_3+J_4, \tag{3.90}$$

with

$$J_3=\int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau|u-v|\frac{|\nabla(u-v)|}{(M-|u-v|)^2},$$
$$J_4=\int_Z b(x)(1+|u|+|v|)^\sigma|\nabla u-\nabla v||\phi(T_{m_1}(u-v))|.$$

Now we will estimate J_3 and J_4. Since $M-|T_{m_1}(u-v)|\ge M-m_1$ almost everywhere on the set $\{|u-v|<m_1\}$ and using property (3.62) we have

$$J_3\le\frac{m_1}{M-m_1}\int_{\{|u-v|<m_1\}\cap Z} c(x)(1+|u|+|v|)^\tau\frac{|\nabla(u-v)|}{M-|u-v|}. \tag{3.91}$$

Due to assumptions (2.24) with arguments already used in Step 1 we obtain that

$$J_3\le\frac{m_1}{M-m_1}S(N,p,t,\Omega,\tau)\|c\|_{L^t(\Omega)}\|1+|\nabla u|+|\nabla v|\|^\tau_{L^p(\Omega)} \times\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}\right)^{1/2} \tag{3.92}$$

where $S(N,p,t,\Omega,\tau)$ is a constant depending on N, p, t Ω and τ.

We now turn to J_4. By property (3.45) of φ, we have

$$|\varphi(T_{m_1}(u-v))|\le\varphi(m_1)\le\frac{m_1}{M(M-m_1)},$$

which leads to

$$J_4 \leq \int_{\{|u-v|<m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma \frac{|\nabla(u-v)|}{M-|u-v|} + \frac{m_1}{M(M-m_1)}\int_{\{|u-v|>m_1\}\cap Z} b(x)(1+|\nabla u|+|\nabla v|)^\sigma |\nabla S_{m_1}(u-v)|. \tag{3.93}$$

Moreover, since $2<N$, assumption on σ implies that

$$\frac{1}{r}+\frac{\sigma}{p}+\frac{1}{2}<1.$$

Therefore we apply Hölder's inequality in (3.93) and we obtain

$$J_4 \leq |\Omega|^{\frac12-\frac1r-\frac{\sigma}{p}}\|b\|_{L^r(\Omega)}\|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}\right)^{1/2} + \frac{m_1}{M(M-m_1)}|\Omega|^{\frac12-\frac1r-\frac{\sigma}{p}}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}\|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)} \times \|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N}. \tag{3.94}$$

On the one hand, since $(1+|\nabla u|+|\nabla v|)^{p-2}\geq 1$, gathering (3.90), (3.92) and (3.94) yields that

$$\beta\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \leq \frac{m_1}{M-m_1}S(N,p,t,\Omega,\tau)\|c\|_{L^t(\Omega)}\|1+|\nabla u|+|\nabla v|\|^\tau_{L^p(\Omega)} \times\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}\right)^{1/2} + |\Omega|^{\frac12-\frac1r-\frac{\sigma}{p}}\|b\|_{L^r(\Omega)}\|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}\right)^{1/2} + \frac{m_1}{M(M-m_1)}|\Omega|^{\frac12-\frac1r-\frac{\sigma}{p}}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)} \times\|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)}\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N}. \tag{3.95}$$

On the other hand by a priori estimate (2.20) given in Remark 4 we know that

$$\|1+|u|+|v|\|^\tau_{L^{p^*}(\Omega)} \quad \text{and} \quad \|1+|\nabla u|+|\nabla v|\|^\sigma_{L^p(\Omega)}$$

are bounded by a constant which depends only on N, $|\Omega|$, p, α, τ, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c.

Therefore by (3.95) we get

$$\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \leq C_1'\frac{m_1}{M-m_1}\|c\|_{L^t(\Omega)}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}\right)^{1/2} + C_1'\|b\|_{L^r(\Omega)}\left(\int_{\{|u-v|<m_1\}}\frac{|\nabla(u-v)|^2}{(M-|u-v|)^2}\right)^{1/2} + \frac{C_1' m_1}{M(M-m_1)}\|b\|_{L^r(\{|u-v|>m_1\}\cap Z)}\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N},$$

where C'_1 is a positive constant which depends only on β, N, $|\Omega|$, p, α, τ, t, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^r(\Omega)}$, but not on c.

Young's inequality, the estimate (3.89) on $\|\nabla S_{m_1}(u-v)\|_{(L^2(\Omega))^N}$ and the definition of m_1 lead to

$$\begin{aligned}\int_{\{|u-v|<m_1\}} \frac{|\nabla(u-v)|^2}{(M-|u-v|)^2} \le C'_2 \left(\frac{m_1}{M-m_1}\right)^2 \|c\|^2_{L^t(\Omega)} \\ + C'_2\|b\|^2_{L^r(\Omega)} + C'_2 \frac{m_1^2}{M(M-m_1)}\|c\|_{L^t(\Omega)},\end{aligned} \tag{3.96}$$

where C'_2 is a positive constant which depends only on β, N, $|\Omega|$, p, α, τ, t, σ, $\|f\|_{W^{-1,p'}(\Omega)}$ and $\|b\|_{L^N(\Omega)}$, but not on c.

Observe that inequality (3.96) coincides with inequality (3.55) in the proof of Theorem 2.1 (in the case $\|b\|_{L^r(\Omega)}$ large and $\|c\|_{L^t(\Omega)}$ small) when p is replaced by 2. With arguments already used in the proof of Theorem 2.1 we conclude that there exists $B' > 0$ such that if $\|c\|_{L^t(\Omega)} \le B'$ then $u = v$ almost everywhere in Ω.

3.3. **Proof of Theorem 2.3.** The proof of Theorem 2.3 under the assumptions of the case i) is exactly the same of the proof of Theorem 2.1 since we have $p < N = 2$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$. The proof of Theorem 2.3 under the assumptions of the case ii) is an adaption of Theorem 2.2 since we have $N = 2 = p$ and therefore we have $p^* = q$ with $1 < q < +\infty$.

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