UNIQUENESS OF THE RENORMALIZED SOLUTION TO A CLASS OF NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We study the uniqueness of the renormalized solution to the elliptic equation $-\text{div}(a(x,u)|Du|^{p-2}Du+\Phi(u)) = f - \text{div}(g)$ in Ω with Dirichlet boundary conditions, where $1 < p \le 2$. We obtain uniqueness of the solution under a weak assumption on $a(x,s)$ and $\Phi(s)$ given under a differential inequality form. It allows us to consider a large class of functions a and Φ with fast growth and/or fast oscillations at infinity.

1. INTRODUCTION

The present paper is concerned with the uniqueness of the solution to the nonlinear elliptic equation

$$\begin{cases} -\text{div}(a(x,u)|Du|^{p-2}Du+\Phi(u)) = f - \text{div}(g) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded an open set of $\mathbb{R}^N$ with $2 \le N$, p is a real number such that $1 < p \le 2$, $a(x,s)$ is a function which satisfies $0 < \alpha \le a(x,s) < +\infty$, Φ is a continuous function from $\mathbb{R}$ into $\mathbb{R}^N$ and $f - \text{div}(g)$ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$.

When the data $f - \text{div}(g)$ belongs to $W^{-1,p'}(\Omega)$ if $a(x,s)$ and $\Phi(s)$ verify a global Lipschitz condition with respect to s (or a global and strong control of the modulus of continuity) uniqueness in the class of weak solutions lying in $W_0^{1,p}(\Omega)$ was obtained in the case where $p = 2$ and $\Phi \equiv 0$ in [1] and for more general operators with $1 < p \le 2$ in [4, 5, 7]. If $p > 2$ the uniqueness of the solution may fail, see e.g. [4, 5] for a counter-example. Partial uniqueness results were recently obtained in [6] when $p > 2$ and $\Phi \equiv 0$.

Without any growth assumption on $a(x,s)$ and $\Phi(s)$ with respect to s we cannot expect to have a solution in the sense of distributions of (1.1) with $L^1 + W^{-1,p'}$ data because of the lack of regularity of the solution. For this reason, here we use the framework of renormalized solutions which insures the existence of a solution to (1.1) without any growth condition on a and Φ. The notion of renormalized solution was introduced in [9, 10] for first order equations and has been developed for elliptic problems with L^1 data in [12] (see also [13]). In [8] the authors give a definition of renormalized solution for elliptic problems with general Radon measure with bounded variation and they prove existence and stability results.

If $p = 2$ and for L^1 data uniqueness results have been obtained in [3] in the framework of renormalized solutions for quasi-linear elliptic problems (with $\Phi \equiv 0$) and in [14] in the very close framework of entropy solutions (the two notions are equivalent for L^1 data). In [14] the modulus of continuity of $a(x,s)$ and Φ have to be controlled by

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$\exp(c|s|)$ $(c>0)$ and in [3] the modulus of continuity of $a(x,s)$ is bounded by a function which verifies an appropriate differential inequality. If the matrix diffusion $A(x,s)$ is locally Hölder continuous in s with a exponent greater or equal to $1/2$ and does not depend on x (or is regular in x) the uniqueness of the renormalized solution is proved in [11] for elliptic equations $-\operatorname{div}(A(x,u)) = f - \operatorname{div}(g)$ with boundary Dirichlet conditions and with $f - \operatorname{div}(g) \in L^1(\Omega) + H^{-1}(\Omega)$.

In the present paper we adapt the uniqueness conditions introduced in [3] to the class of elliptic equations of the type (1.1) with $1 < p \le 2$. We state in Theorem 2.4 that if the modulus of continuity of $a(x,s)$ and $\Phi(s)$ are bounded by a function which verifies a differential inequality then the renormalized solution of (1.1) is unique. This result seems to be new even if the data $f - \operatorname{div}(g)$ belongs to $W^{-1,p'}(\Omega)$. As an example, if $b(x)$ is a non negative element of $L^\infty(\Omega)$ then Theorem 2.4 insures the uniqueness of the renormalized solution to the equation

$$\begin{cases} -\operatorname{div}((1+b(x)\exp(\exp(u^2)))|Du|^{p-2}Du + \exp(u^4)S) = f - \operatorname{div}(g) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $S \in \mathbb{R}^N$.

Let us consider two solutions u and v of the equation $-\operatorname{div}(a(x,u)Du) = f$ with boundary Dirichlet conditions. If $f \in L^2(\Omega)$ the method developed in [1, 4, 5, 7] consists in using the test function $T_K(u-v)$ against the difference of the two equations, where T_K is the truncation function at height $\pm K$. It follows that

$$\int_\Omega a(x,u)|DT_K(u-v)|^2 dx = \int_\Omega (a(x,v) - a(x,u))Dv \cdot DT_K(u-v)dx.$$

Since the weak solution belongs to $H_0^1(\Omega)$ the Lipschitz condition on $a(x,s)$ with respect to s allows to conclude the proof by dividing the equality by K^2 and passing to the limit as K goes to zero (with the help of Poincaré inequality). If f belongs to $L^1(\Omega)$ this method fails: the main obstacle is the lack of regularity, $(a(x,v) - a(x,u))Dv$ cannot be expected in $L^2(\Omega)$ if $a(x,s)$ is global Lipschitz with respect to s. In the present paper (see also [3]) we use $T_K(\varphi(u) - \varphi(v))$ in place of $T_K(u-v)$, where φ is an appropriate function. Notice that $T_K(\varphi(u) - \varphi(v))$ is not an admissible test function. Formally, a few computations leads to

$$\int_\Omega \frac{a(x,u)}{\varphi'(u)}|DT_K(\varphi(u) - \varphi(v))|^2 dx \\ = \int_\Omega \Big(\frac{a(x,v)}{\varphi'(v)} - \frac{a(x,u)}{\varphi'(u)}\Big) D\varphi(v) \cdot DT_K(\varphi(u) - \varphi(v))dx.$$

In Theorem 3.1 we give fairly technical assumptions on a and φ which insure that

$$1\!\!1_{\{|\varphi(u)-\varphi(v)|<K\}} \Big(\frac{a(x,v)}{\varphi'(v)} - \frac{a(x,u)}{\varphi'(u)}\Big)(\varphi'(u))^{1/2} D\varphi(v) \in (L^2(\Omega))^N,$$

and allow us to continue the process. In Theorem 2.4 we prove that if the modulus of continuity in s of $a(x,s)$ is bounded by a function verifying an appropriate differential inequality then we are able to construct a function φ which fulfils the conditions of Theorem 3.1.

Finally let us mention that the question of the uniqueness under a very local condition in s and/or $p > 2$ remains still open in general.

The paper is organized as follows: Section 2 is devoted to give the assumptions on the data and to recall the definition of a renormalized solution to equation (1.1). Then we state the main uniqueness result in Theorem 2.4. In Section 3 we prove Theorem 3.1 which insures the uniqueness of the solution under technical assumptions.

2. Definitions and main result

In the whole paper we assume that Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 2$) and p is a real number such that $1 < p \leq 2$. The function $a : \Omega \times \mathbb{R} \mapsto \mathbb{R}^+$ is a Carathéodory function which verifies

$$\exists \alpha > 0, \forall K > 0, \exists C_K > 0, \quad \alpha \leq a(x,s) \leq C_K \quad \forall |s| \leq K, \text{ a.e. in } \Omega; \tag{2.1}$$

$$\text{the function } \Phi : \mathbb{R} \mapsto \mathbb{R}^N \text{ is continuous;} \tag{2.2}$$

$$f \in L^1(\Omega), \quad g \in (L^{p'}(\Omega))^N. \tag{2.3}$$

For any $K > 0$ we denote by T_K the truncation function at height $\pm K$, i.e. $T_K(s) = \max(-K, \min(K,s))$ for any $s \in \mathbb{R}$ and we define the continuous function h_n by

$$h_n(s) = 1 - \Big| \frac{T_{2n}(s) - T_n(s)}{n} \Big|. \tag{2.4}$$

We now recall the definition of the gradient of functions whose truncates belong to $W_0^{1,p}(\Omega)$ (see [2]) and the definition of a renormalized solution to Equation (1.1) (see [8, 12, 13]).

Definition 2.1. Let $u : \Omega \mapsto \mathbb{R}$ be a measurable function, finite almost everywhere in Ω, such that $T_K(u) \in W_0^{1,p}(\Omega)$ for any $K > 0$. Then there exists a unique measurable vector field $v : \Omega \mapsto \mathbb{R}^N$ such that

$$DT_K(u) = \mathbb{1}_{\{|u|<K\}} v \quad \text{a.e. in } \Omega.$$

This function v is called the gradient of u and is denoted by Du.

Definition 2.2. A measurable function u defined from Ω into $\mathbb{R}$ is a renormalized solution of (1.1) if

$$\forall K > 0, \quad T_K(u) \in W_0^{1,p}(\Omega); \tag{2.5}$$

if for any function $h \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} h$ is compact, u satisfies the equation

$$\begin{aligned} -\operatorname{div}\big[h(u)a(x,u)|Du|^{p-2}Du + h(u)\Phi(u)\big] + h'(u)a(x,u)|Du|^p \\ + h'(u)\Phi(u)\cdot Du = fh(u) - \operatorname{div}(gh(u)) + h'(u)g\cdot Du \quad \text{in } \mathscr{D}'(\Omega), \end{aligned} \tag{2.6}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{|u|<n\}} a(x,u)|Du|^p\, dx = 0. \tag{2.7}$$

Remark 2.3. It is well known that under conditions (2.1)–(2.3) there exists at least one renormalized solutions of (1.1) (see e.g. [8, 12, 13]).

Condition (2.5) and Definition 2.1 allow to define Du almost everywhere in Ω. Equation (2.6) is formally obtained by the point-wise multiplication of (1.1) by $h(u)$ and every terms are well defined. Indeed since $\operatorname{supp}(h)$ is compact, we have $\operatorname{supp}(h) \subset [-K,K]$ for $K > 0$ sufficiently large. It follows that

$$h(u)a(x,u)|Du|^{p-2}Du = h(u)a(x,T_K(u))|DT_K(u)|^{p-2}DT_K(u)$$

almost everywhere in Ω and then it belongs to $(L^{p'}(\Omega))^N$. Similarly $h'(u)a(x,u)|Du|^p$ is identified to $h'(u)a(x,T_K(u))|DT_K(u)|^p$ which belongs to $L^1(\Omega)$ and it is immediate that $h(u)\Phi(u)\in(L^\infty(\Omega))^N$ and $h'(u)\Phi(u)\cdot Du\in L^p(\Omega)$. The same arguments imply that the right hand side of (2.6) lies in $L^1(\Omega)+W^{-1,p'}(\Omega)$. Condition (2.7) is classical in the framework of renormalized solutions and gives additional information on the behavior of Du for large value of $|u|$.

In the following theorem we claim that if the modulus of continuity of the functions $a(x,\cdot)$ and $\Phi(\cdot)$ are uniformly controlled by a function which verifies a differential inequality then the uniqueness of the renormalized solution holds.

Theorem 2.4. *Assume that (2.1)–(2.3) hold true and that there exists a function* $w\in\mathscr{C}^1(\mathbb{R})$ *such that*

$$w\geq 0, \tag{2.8}$$

$$\exists\eta>0,\exists C_1>0\quad |w'|\leq C_1 w^{1+\eta}, \tag{2.9}$$

$$|a(x,s)-a(x,t)|\leq\left|\int_s^t w(z)dz\right|,\quad\forall s,t\in\mathbb{R},\text{ a.e. in }\Omega, \tag{2.10}$$

$$|\Phi(s)-\Phi(t)|\leq\left|\int_s^t w(z)dz\right|,\quad\forall s,t\in\mathbb{R}. \tag{2.11}$$

Then the renormalized solution of (1.1) is unique.

Remark 2.5. Conditions (2.10) and (2.11) impose global properties on the function a and Φ. Hypothesis (2.9) allows a large class of functions w: polynomial, exponential functions, exponential of polynomial functions,... are dominated by a function verifying (2.8) and (2.9). For example if b is a non negative function belonging to $L^\infty(\Omega)$, the function $a(x,s)=1+b(x)\sin\big(\exp(\exp(s^2))\big)$ verifies assumptions (2.8)–(2.10). We can have highly oscillating or/and increasing $a(x,s)$ and $\Phi(s)$ with respect to s. The reader may convince himself that there exists for example a function Φ which is local Lipschitz continuous and which does not verify (2.8), (2.9) and (2.11).

3. Proof of Theorem 2.4

The proof of Theorem 2.4 relies on a technical result which is a generalization of Theorem 3.2 established in [3] in the case of quasi-linear degenerate elliptic problems.

Theorem 3.1. *Assume that (2.1)–(2.3) hold true and that there exists a real valued function* φ *belonging to* $\mathscr{C}^1(\mathbb{R})$ *such that:*

$$\varphi(0)=0,\quad\varphi'\geq 1. \tag{3.1}$$

Moreover assume that there exists $\delta>1/2$, $K_0>0$ *and* $L>0$ *such that*

$$\frac{\varphi'}{(1+|\varphi|)^{2\delta}}\in L^\infty(\mathbb{R}); \tag{3.2}$$

for any $r, s \in \mathbb{R}$ and any $0 < K \le K_0$ such that $|\varphi(r) - \varphi(s)| \le K$ we have

$$\left| \frac{a(x,r)}{(\varphi'(r))^{p-1}} - \frac{a(x,s)}{(\varphi'(s))^{p-1}} \right| \le \frac{1}{(\varphi'(s))^{p-1}} \times \frac{LK}{\big(1+|\varphi(r)|+|\varphi(s)|\big)^{\delta}} \text{ a.e. in } \Omega, \tag{3.3}$$

$$\Big|\Phi(r) - \Phi(s)\Big| \le \frac{1}{(\varphi'(s))^{(p-1)/p}} \times \frac{LK}{\big(1+|\varphi(r)|+|\varphi(s)|\big)^{(2-p)\delta/p}}, \tag{3.4}$$

$$\frac{1}{L} \le \frac{\varphi'(s)}{\varphi'(r)} \le L. \tag{3.5}$$

Then the renormalized solution of (1.1) is unique.

Remark 3.2. Assumption (3.2) allows to deal with the $W^{-1,p'}(\Omega)$ part of the right hand side of (1.1) and it is important to obtain estimate (3.6) in Proposition 3.4 below. If we assume that $f - \text{div}(g) \in L^1(\Omega) + W^{-1,p'}(\Omega)$ is also a Radon measure of bounded variation on Ω then assumption (3.2) is not necessary (see Remark 3.5 below).

Remark 3.3. The methods used for the proofs of Theorem 2.4 and 3.1 can be adapted to obtain a comparison result. Moreover if we replace the operator $-\text{div}(a(x,u)|Du|^{p-2}Du)$ with more general operators in the form $-\text{div}(a(x,u,Du))$ (with $1 < p \le 2$) it is possible to give fairly technical assumptions in order to obtain similar results.

We state in the following proposition an estimate of any renormalized solution of (1.1). Such an estimate is classical in the framework of renormalized solution. For the convenience of the reader we give the proof.

Proposition 3.4. *Let u be a renormalized solution of (1.1). If φ is a bounded and increasing function belonging to $\mathscr{C}^1$ such that φ' is bounded and $\varphi(0) = 0$ then u satisfies*

$$\varphi'(u)a(x,u)|Du|^p \in L^1(\Omega). \tag{3.6}$$

Proof of Proposition 3.4. The method is standard: we take $h = h_n$ and $\varphi(T_{2n}(u))$ which belongs to $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ as a test function in (2.6) and we pass to the limit as n tends to infinity. We observe that $h_n(u)\varphi(T_{2n}(u)) = h_n(u)\varphi(u)$ almost everywhere in Ω. Moreover due to the divergence theorem the contribution of $-\text{div}(h_n(u)\Phi(u))$ and $h_n'(u)\Phi(u) \cdot Du$ against the test function $\varphi(T_{2n}(u))$ is equal to zero. It follows that

$$\begin{aligned} &\int_\Omega h_n(u)a(x,u)|Du|^{p-2}Du \cdot D\varphi(u)\,dx + \int_\Omega h_n'(u)\varphi(u)a(x,u)|Du|^p\,dx \\ &\qquad = \int_\Omega h_n(u)f\varphi(u)\,dx + \int_\Omega h_n'(u)\varphi(u)g \cdot Du\,dx + \int_\Omega h_n(u)g \cdot D\varphi(u)\,dx. \end{aligned} \tag{3.7}$$

Since $0 \le h_n(s) \le 1$ $\forall s \in \mathbb{R}$ and since φ is bounded we have

$$\left| \int_\Omega h_n(u)f\varphi(u)\,dx \right| \le \|\varphi\|_{L^\infty(\mathbb{R})} \|f\|_{L^1(\Omega)}.$$

Condition (2.7) implies (because φ is bounded)

$$\lim_{n\to\infty} \int_\Omega h_n'(u)\varphi(u)a(x,u)|Du|^p\,dx = 0,$$

and using (2.1), (2.7) and Hölder's inequality we also have

$$\lim_{n\to\infty} \int_\Omega h_n'(u)\varphi(u)g \cdot Du\,dx = 0.$$

Concerning the last term in the right hand side of (3.7) because the functions h_n and φ' are non negative and bounded, Young inequality yields that for any $\varepsilon > 0$

$$\left|\int_\Omega h_n(u) g\cdot D\varphi(u)\,dx\right| \le C(p,\varepsilon)\|\varphi'\|_{L^\infty(\mathbb{R})}\int_\Omega |g|^{p'}dx + \varepsilon\int_\Omega h_n(u)\varphi'(u)|Du|^p dx, \tag{3.8}$$

where $C(p,\varepsilon) > 0$ is a constant depending only on p and ε.

From assumption (2.1) and (3.6) choosing $\varepsilon = \delta/2$ in the above inequality gives

$$\begin{aligned}\int_\Omega h_n(u)\varphi'(u)a(x,u)|Du|^p dx \\ \le C(p,\delta)\Big(\|\varphi'\|_{L^\infty(\mathbb{R})}\int_\Omega |g|^{p'}dx + \|\varphi\|_{L^\infty(\mathbb{R})}\|f\|_{L^1(\Omega)}\Big) + \omega(n),\end{aligned}$$

where $\omega(n)$ tends to zero as n tends to infinity.

Since $\varphi'(u)a(x,u)|Du|^p \ge 0$ and since $h_n(u)$ tends to 1 almost everywhere in Ω as n tends to infinity Fatou's lemma allows us to complete the proof. ☐

Remark 3.5. The fact that φ' is bounded is only needed in inequality (3.8). Indeed if we assume that $f - \operatorname{div}(g)$ is also a Radon measure of bounded variation on Ω we have

$$\left|\int_\Omega f h_n(u)\varphi\,dx + \int_\Omega g\cdot D(h_n(u)\varphi)\,dx\right| \le M\|\varphi\|_{L^\infty(\Omega)}$$

where $M > 0$ is constant depending on f and g, so that the condition φ' bounded is not required in this particular case.

Proof of Theorem 3.1. The proof will be divided into two steps. Let u and v be two renormalized solutions of (1.1). Step 1 is devoted to show that

$$\lim_{K\to 0}\frac{1}{K^2}\int_\Omega\Big(\frac{1}{(\varphi'(u))^{p-1}} + \frac{1}{(\varphi'(v))^{p-1}}\Big)\frac{|DT_K(\varphi(u) - D\varphi(v))|^2}{(|D\varphi(u)| + |D\varphi(v)|)^{2-p}}dx = 0.$$

Notice that the regularity of u and v do not imply, in general, that the previous integral exists for any fixed $K > 0$. In Step 2 we use this result to show that $u = v$ almost everywhere in Ω.

Step 1. Writing the renormalized formulation (2.6) for u with h_n in place of h gives

$$\begin{aligned}-\operatorname{div}\big[h_n(u)a(x,u)|Du|^{p-2}Du + h_n(u)\Phi(u)\big] + h_n'(u)a(x,u)|Du|^p \\ + h_n'(u)\Phi(u)\cdot Du = f h_n(u) - \operatorname{div}(g h_n(u)) + h_n'(u) g\cdot Du \quad \text{in } \mathscr{D}'(\Omega).\end{aligned}\tag{3.9}$$

Since Φ is local Lipschitz continuous and since h_n has a compact support, the regularity of $T_k(u)$ yields that

$$\operatorname{div}\big(h_n(u)\Phi(u)\big) - h_n'(u)\Phi(u)\cdot Du = \operatorname{div}\big(\Phi_n(T_{2n+1}(u))\big) = \operatorname{div}\big(\Phi_n(u)\big), \tag{3.10}$$

where $\Phi_n = \big((\Phi_n)_1,\dots,(\Phi_n)_N\big)$ is defined by

$$\big(\Phi_n\big)_i(r) = \int_0^r \Phi_i'(t)h_n(t)\,dt. \tag{3.11}$$

Let us consider the function $W_K = T_K\big(\varphi(u_{3n}) - \varphi(v_{3n})\big)$ where u_{3n} and v_{3n} denote $T_{3n}(u)$ and $T_{3n}(v)$ respectively. From (2.5) it follows that W_K belongs to $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ and then we can use W_K as a test function in (3.9) written for both u and v. By subtracting the two equalities we obtain that

$$A_{K,n}+B_{K,n}+C_{K,n}=D_{K,n}+E_{K,n}+F_{K,n} \tag{3.12}$$

where

$$\begin{aligned}
A_{K,n}&=\int_\Omega \big(h_n(u)a(x,u)|Du|^{p-2}Du-h_n(v)a(x,v)|Dv|^{p-2}Dv\big)\cdot DW_K\,dx,\\
B_{K,n}&=\int_\Omega \big(h_n'(u)a(x,u)|Du|^{p}-h_n'(v)a(x,v)|Dv|^{p}\big)W_K\,dx,\\
C_{K,n}&=\int_\Omega \big(\Phi_n(u)-\Phi_n(v)\big)\cdot DW_k\,dx,\\
D_{K,n}&=\int_\Omega f(h_n(u)-h_n(v))W_K\,dx,\\
E_{K,n}&=\int_\Omega \big(h_n(u)-h_n(v)\big)g\cdot DW_K\,dx,\\
F_{K,n}&=\int_\Omega g\cdot\big(h_n'(u)Du-h_n'(v)Dv\big)W_K\,dx.
\end{aligned}$$

We now pass to the limit in (3.12) first as n goes to infinity first and then as K goes to zero. To shorten the notations we denote $U_K=\{x\in\Omega\,;\,0<|\varphi(u)-\varphi(v)|<K\}$. We now study all the terms in (3.12).

We first write $A_{K,n}$ as

$$\begin{aligned}
A_{K,n}=&\int_\Omega h_n(u)\frac{a(x,u)}{(\varphi'(u))^{p-1}}\big(|D\varphi(u)|^{p-2}D\varphi(u)-|D\varphi(v)|^{p-2}D\varphi(v)\big)\cdot DW_K\,dx\\
&+\int_\Omega h_n(u)\Big(\frac{a(x,u)}{(\varphi'(u))^{p-1}}-\frac{a(x,v)}{(\varphi'(v))^{p-1}}\Big)(\varphi'(v))^{p-1}|Dv|^{p-1}Dv\cdot DW_K\,dx\\
&+\int_\Omega \big(h_n(u)-h_n(v)\big)a(x,v)|Dv|^{p-2}Dv\cdot DW_K\,dx.
\end{aligned}$$

By symmetry with respect to v we obtain the following decomposition

$$\begin{aligned}
A_{K,n}=&\frac12\int_\Omega\Big(h_n(u)\frac{a(x,u)}{(\varphi'(u))^{p-1}}+h_n(v)\frac{a(x,v)}{(\varphi'(v))^{p-1}}\Big)\\
&\times\big(|D\varphi(u)|^{p-2}D\varphi(u)-|D\varphi(v)|^{p-2}D\varphi(v)\big)\cdot DW_K\,dx\\
&+\frac12\int_\Omega\Big(\frac{a(x,u)}{(\varphi'(u))^{p-1}}-\frac{a(x,v)}{(\varphi'(v))^{p-1}}\Big)\\
&\times\big(h_n(u)(\varphi'(v))^{p-1}|Dv|^{p-2}Dv+h_n(v)(\varphi'(u))^{p-1}|Du|^{p-2}Du\big)\cdot DW_K\,dx\\
&+\frac12\int_\Omega\big(h_n(u)-h_n(v)\big)\big(a(x,u)|Du|^{p-2}Du+a(x,v)|Dv|^{p-2}Dv\big)\cdot DW_K\,dx
\end{aligned}$$

which reads as

$$A_{K,n}=A^1_{K,n}+A^2_{K,n}+A^3_{K,n}. \tag{3.13}$$

We remark that since $\varphi'\ge1$ we have $|u_{3n}-v_{3n}|\le|\varphi(u_{3n})-\varphi(v_{3n})|$ almost everywhere in Ω. Recalling the definition of W_K and that the support of h_n is $[-2n,2n]$ we deduce that for K small enough

$$h_n(u)DT_K(\varphi(u)-\varphi(v))=h_n(u)DT_K\big(\varphi(u_{3n})-\varphi(v_{3n})\big)=h_n(u)DW_K, \tag{3.14}$$

and the same equality with v in place of u.

Using (2.1), (3.14) and the strong monotonicity of $|\xi|^{p-2}\xi$ we obtain

$$(p-1)\alpha\int_{U_K}\Big(\frac{h_n(u)}{(\varphi'(u))^{p-1}}+\frac{h_n(v)}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx\le A^1_{K,n}. \tag{3.15}$$

For any $0<K\le K_0$, assumptions (3.3) and (3.5) together with (3.14) and Young inequality lead to, with a few computations,

$$\begin{aligned}|A^2_{K,n}|\le & CK^2\int_{U_K}\frac{1}{(\varphi'(u))^{2(p-1)}}\frac{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}{\big(1+|\varphi(u)|+|\varphi(v)|\big)^{2\delta}}\\ &\times\big(h_n(u)\varphi(v)^{p-1}|D\varphi(v)|^{2p-2}+h_n(v)\varphi(u)^{p-1}|D\varphi(u)|^{2p-2}\big)dx\\ &+(p-1)\frac{\alpha}{10}\int_{U_K}\Big(\frac{h_n(u)}{(\varphi'(u))^{p-1}}+\frac{h_n(v)}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx,\end{aligned}$$

where C is a generic constant which depends on p, Ω, L, α and does not depend on K.

Using again (3.5) which implies that $\frac{1}{L}\le\frac{\varphi'(u)}{\varphi'(v)}\le L$ almost everywhere on U_K the first term in the above inequality can be simplified and we obtain that

$$\begin{aligned}|A^2_{K,n}|\le & CK^2\int_{U_K}(h_n(u)+h_n(v))\frac{\varphi'(u)|Du|^p+\varphi'(v)|Dv|^p}{\big(1+|\varphi(u)|+|\varphi(v)|\big)^{2\delta}}dx\\ &+(p-1)\frac{\alpha}{10}\int_{U_K}\Big(\frac{h_n(u)}{(\varphi'(u))^{p-1}}+\frac{h_n(v)}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx.\end{aligned} \tag{3.16}$$

As far as $A^3_{K,n}$ is concerned since φ is an increasing function assumption (3.5) implies that

$$|u-v|\le\frac{L}{\varphi'(u)}|\varphi(u)-\varphi(v)|\quad\text{a.e. in } U_K. \tag{3.17}$$

Since h_n is Lipschitz continuous with $|h'_n|\le 1/n$ almost everywhere in $\mathbb{R}$ while $\operatorname{supp}(h_n)=[-2n,2n]$ we obtain for $0<K\le K_0$ and K small enough

$$\begin{aligned}|A^3_{K,n}|\le&\frac{CK}{n}\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}\frac{1}{\varphi'(u)}\big(\varphi'(u)a(x,u)|Du|^p+\varphi'(v)a(x,v)|Dv|^p\\ &\qquad+\varphi'(u)a(x,u)|Du|^{p-1}|Dv|+\varphi'(v)a(x,v)|Dv||Du|dx\big)dx\\ \le&\frac{CK}{n}\int_{\substack{\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}\big(a(x,u)|Du|^p+a(x,v)|Dv|^p\big)dx\\ &+\frac{CK}{n}\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}\big(a(x,u)|Dv|^p+a(x,v)|Dv|^p\big)dx.\end{aligned}$$

Due to (2.7) the first term in the previous inequality goes to zero as n goes to infinity. We now claim that

$$\frac{CK}{n}\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}a(x,u)|Dv|^p dx\to 0\text{ as }n\to\infty. \tag{3.18}$$

Remark that since the function $a(x,s)$ is not assumed to be bounded (3.18) is not a direct consequence of the energy condition (2.7): we also use assumptions (3.3) and (3.5).

Indeed we first write

$$\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} a(x,u)|Dv|^p\,dx = \int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} \Big(\frac{a(x,u)}{(\varphi'(u))^{p-1}} - \frac{a(x,v)}{(\varphi'(v))^{p-1}}\Big)(\varphi'(u))^{p-1}|Dv|^p\,dx + \int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} \frac{a(x,v)}{(\varphi'(v))^{p-1}}(\varphi'(u))^{p-1}|Dv|^p\,dx.$$

Using assumptions (3.3) and (3.5) we deduce that

$$\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} a(x,u)|Dv|^p\,dx \le LK\int_{\{|v|<2n+1\}} \frac{|Dv|^p}{(1+|\varphi(u)|+|\varphi(v)|)^\delta}\,dx + L\int_{\{|v|<2n+1\}} a(x,v)|Dv|^p\,dx$$

and together with (2.1) and (2.7) we obtain (3.18). We conclude that

$$\lim_{n\to\infty}|A^3_{K,n}| = 0. \tag{3.19}$$

We now turn to $B_{K,n}$. Since W_K belongs to $L^\infty(\Omega)$ with $\|W_K\|_{L^\infty(\Omega)} \le K$ we have

$$|B_{K,n}| \le \frac{K}{n}\int_{\{|u|<2n\}} a(x,u)|Du|^p\,dx + \frac{K}{n}\int_{\{|v|<2n\}} a(x,v)|Dv|^p\,dx.$$

From (2.7) it follows that

$$\lim_{n\to\infty}|B_{K,n}| = 0. \tag{3.20}$$

As far as $C_{K,n}$ is concerned (3.10) and (3.11) allow us to write

$$C_{K,n} = \int_\Omega \Big(\int_v^u \Phi'(s)h_n(s)\,ds\Big)\cdot DT_K\big(\varphi(u_{3n})-\varphi(v_{3n})\big)\,dx.$$

Since the function φ is increasing with $\varphi' \ge 1$, for K small enough we have

$$\mathbb{1}_{\{|v|<2n\}}DT_K\big(\varphi(u_{3n})-\varphi(v_{3n})\big) = \mathbb{1}_{\{|v|<2n\}}DT_K\big(\varphi(u)-\varphi(v)\big)$$

almost everywhere on $\{|u|<2n+1\}$. Recalling that the support of h_n is $[-2n,2n]$ the integral $C_{K,n}$ reads as

$$C_{K,n} = \int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} \Big(\int_v^u \Phi'(s)h_n(s)\,ds\Big)\cdot DT_K(\varphi(u)-\varphi(v))\,dx.$$

Since h_n is increasing on $[-2n,-n]$ and decreasing on $[n,2n]$ while $h_n(s)=1$ on $[-n,n]$, for similar reasons if K is small enough and if $r<t$ are such that $|\varphi(r)-\varphi(t)|<K$ we have

$$\forall s\in[r,t] \quad h_n(s) \le h_n(r)+h_n(t).$$

It follows that

$$|C_{K,n}| \le \int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} (h_n(u)+h_n(v))|\Phi(u)-\Phi(v)||D(\varphi(u)-\varphi(v))|\,dx.$$

Young inequality leads to

$$\begin{aligned}|C_{K,n}| \leq & C\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}} |\Phi(u)-\Phi(v)|^2\big(|D\varphi(u)|+|D\varphi(v)|\big)^{2-p} \\ & \qquad\times\big(h_n(u)\varphi'(u)^{p-1}+h_n(v)(\varphi'(v))^{p-1}\big)dx \\ & +\frac{(p-1)\delta}{10}\int_{U_K}\Big(\frac{h_n(u)}{(\varphi'(u))^{p-1}}+\frac{h_n(v)}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx.\end{aligned} \tag{3.21}$$

Using assumption (3.4) on Φ and (3.5) we have for K small enough

$$\begin{aligned}&\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}|\Phi(u)-\Phi(v)|^2\big(|D\varphi(u)|+|D\varphi(v)|\big)^{2-p}\big(h_n(u)\varphi'(u)^{p-1}+h_n(v)(\varphi'(v))^{p-1}\big)dx \\ &\qquad\leq CK^2\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}\frac{1}{(\varphi'(u))^{2(p-1)/p}}\times\frac{\varphi'(u)|Du|^{2-p}+\varphi'(v)|Dv|^{2-p}}{(1+|\varphi(u)|+|\varphi(v)|)^{2\delta(2-p)/p}}dx.\end{aligned}$$

Using Hölder's inequality, (3.5) and (3.21) imply that

$$\begin{aligned}|C_{K,n}| \leq & CK^2\Big(\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}\frac{\varphi'(u)|Du|^p+\varphi'(v)|Dv|^p}{(1+|\varphi(u)|+|\varphi(v)|)^{2\delta}}dx\Big)^{(2-p)/p} \\ & +\frac{(p-1)\delta}{10}\int_{U_K}\Big(\frac{h_n(u)}{(\varphi'(u))^{p-1}}+\frac{h_n(v)}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx.\end{aligned} \tag{3.22}$$

We now prove that the terms $D_{K,n}$, $E_{K,n}$, $F_{K,n}$ of the right hand side of (3.12) go to zero as n goes to infinity. Since the fields u and v are finite almost everywhere in Ω the function $|h_n(u)-h_n(v)|$ goes to zero as n goes to infinity (h_n converges to 1) while it is bounded by 1. Recalling that f belongs to $L^1(\Omega)$ and $\|W_K\|_{L^\infty(\Omega)}\leq K$ the Lebesgue dominated convergence theorem gives that

$$D_{K,n}=\int_\Omega (h_n(u)-h_n(v))fW_K\,dx\to 0\quad\text{as } n\to\infty.$$

For $E_{K,n}$, recalling (3.17) and that h_n is Lipschitz continuous we have for K small enough

$$\begin{aligned}|E_{K,n}| \leq & \frac{LK}{n}\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}|g|\frac{\varphi'(u)|Du|+\varphi'(v)|Dv|}{\varphi'(u)}dx \\ \leq & \frac{CK}{n}\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}|g|\big(|Du|+|Dv|\big)dx \\ \leq & \frac{CK}{n}\|g\|_{(L^{p'}(\Omega))^N}\Big(\int_{\{|u|<2n+1\}}|Du|^p dx+\int_{\{|v|<2n+1\}}|Dv|^p dx\Big)^{1/p}.\end{aligned}$$

The energy condition (2.7) allows us to conclude that

$$\lim_{n\to\infty} E_{K,n}=0. \tag{3.23}$$

As far as $F_{K,n}$ is concerned and since $|W_K|\leq K$ Hölder's inequality yields that

$$|F_{K,n}|\leq\frac{2K}{n}\Big(\int_\Omega|g|^{p'}dx+\int_\Omega|Du|^p dx+\int_\Omega|Dv|^p dx\Big)$$

and (2.7) allows us to conclude that

$$\lim_{n\to\infty} F_{K,n}=0. \tag{3.24}$$

Gathering (3.12), (3.13), (3.15), (3.16), (3.19), (3.20), (3.22)–(3.24) leads to

$$\begin{aligned}
&\int_{U_K}\Big(\frac{h_n(u)}{(\varphi'(u))^{p-1}}+\frac{h_n(v)}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}\,dx\\
&\quad\le CK^2\int_{U_K}(h_n(u)+h_n(v))\frac{\varphi'(u)|Du|^p+\varphi'(v)|Dv|^p}{\big(1+|\varphi(u)|+|\varphi(v)|\big)^{2\delta}}\,dx\\
&\qquad+CK^2\Big(\int_{\substack{U_k\cap\{|u|<2n+1\}\\ \cap\{|v|<2n+1\}}}\frac{\varphi'(u)|Du|^p+\varphi'(v)|Dv|^p}{(1+|\varphi(u)|+|\varphi(v)|)^{2\delta}}\,dx\Big)^{(2-p)/p}+\omega(n),
\end{aligned}\tag{3.25}$$

where C is a constant depending only on p, δ, Ω and L and where $\omega(n)$ goes to zero as n goes to infinity.

Since $2\delta>1$, we can apply Proposition 3.4 to the function $r\mapsto\int_0^r\frac{\varphi'(s)}{(1+|\varphi(s)|)^{2\delta}}\,ds$ and we obtain that

$$\frac{\varphi'(u)|Du|^p}{\big(1+|\varphi(u)|\big)^{2\delta}}\in L^1(\Omega)\quad\text{and}\quad\frac{\varphi'(v)|Dv|^p}{\big(1+|\varphi(v)|\big)^{2\delta}}\in L^1(\Omega).\tag{3.26}$$

We are now in a position to pass to the limit as n goes to infinity in the right hand side of (3.25): Fatou's lemma implies that

$$\begin{aligned}
&\int_{U_K}\Big(\frac{1}{(\varphi'(u))^{p-1}}+\frac{1}{(\varphi'(v))^{p-1}}\Big)\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}\,dx\\
&\quad\le CK^2\int_{U_K}\frac{\varphi'(u)|Du|^p+\varphi'(v)|Dv|^p}{\big(1+|\varphi(u)|+|\varphi(v)|\big)^{2\delta}}\,dx\\
&\qquad+CK^2\Big(\int_{U_k}\frac{\varphi'(u)|Du|^p+\varphi'(v)|Dv|^p}{(1+|\varphi(u)|+|\varphi(v)|)^{2\delta}}\,dx\Big)^{(2-p)/p}.
\end{aligned}$$

Since $U_K=\{0<|\varphi(u)-\varphi(v)|<K\}$ the function $1\!\!1_{U_K}$ tends to zero almost everywhere in Ω as K tends to zero. Thus dividing the above inequality by K^2, the Lebesgue dominated convergence theorem and (3.26) allow us to conclude that

$$\lim_{K\to0}\frac{1}{K^2}\int_\Omega\Big(\frac{1}{(\varphi'(u))^{p-1}}+\frac{1}{(\varphi'(v))^{p-1}}\Big)\frac{|DT_K(\varphi(u)-D\varphi(v))|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}\,dx=0.\tag{3.27}$$

Step 2. We now prove that (2.7) and (3.27) imply that $u=v$ almost everywhere in Ω. We first consider the function $h_n(u)T_K(\varphi(u)-\varphi(v))$ which belongs to $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ because of (2.5) and the definition of h_n. Poincaré inequality gives

$$\begin{aligned}
\int_\Omega(h_n(u))^p\Big|\frac{T_K(\varphi(u)-\varphi(v))}{K}\Big|^p dx\le C\Big(&\frac{1}{n^p}\int_{\{|u|<2n\}}|Du|^p\Big|\frac{T_K(\varphi(u)-\varphi(v))}{K}\Big|^p dx\\
&+\int_\Omega(h_n(u))^p\frac{|DT_K(\varphi(u)-\varphi(v))|^p}{K^p}\,dx\Big).
\end{aligned}\tag{3.28}$$

We first let K tends to zero and then n tends to infinity in the above inequality. Since for any K we have

$$\frac{1}{n^p}\int_{\{|u|<2n\}}|Du|^p\Big|\frac{T_K(\varphi(u)-\varphi(v))}{K}\Big|^p dx\le\frac{1}{n^p}\int_{\{|u|<2n\}}|Du|^p\,dx$$

the energy condition (2.7) implies that

$$\lim_{n\to\infty}\limsup_{K\to 0}\frac{1}{n^p}\int_{\{|u|<2n\}}|Du|^p\Big|\frac{T_K(\varphi(u)-\varphi(v))}{K}\Big|^p dx=0. \tag{3.29}$$

Concerning the second term in the right hand side of (3.28), if $1<p<2$ Hölder's inequality and the regularity of u and v give for K small enough

$$\begin{aligned}
&\int_\Omega (h_n(u))^p\frac{|DT_K(\varphi(u)-\varphi(v))|^p}{K^p}dx\\
&\le\Big(\frac{1}{K^2}\int_{U_K\cap\{|u|<2n\}}\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx\Big)^{p/2}\\
&\quad\times\Big(\int_{U_K\cap\{|u|<2n\}}\big(|D\varphi(u)|+|D\varphi(v)|\big)^p dx\Big)^{(2-p)/2}\\
&\le\Bigg(\frac{\max\limits_{s\in[-2n,2n]}(\varphi'(s))^{p-1}}{K^2}\int_{U_K\cap\{|u|<2n\}}\frac{1}{(\varphi'(u))^{p-1}}\times\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx\Bigg)^{p/2}\\
&\quad\times\Big(\int_{\substack{\{|u|<2n\}\\ \{|v|<3n\}}}\varphi'(u)^p|Du|^p+\varphi'(v)^p|Dv|^p dx\Big)^{(2-p)/2}\\
&\le C(n,u,v)\Big(\frac{1}{K^2}\int_{U_K}\frac{1}{(\varphi'(u))^{p-1}}\times\frac{|D\varphi(u)-D\varphi(v)|^2}{(|D\varphi(u)|+|D\varphi(v)|)^{2-p}}dx\Big)^{p/2}
\end{aligned}$$

where $C(n,u,v)$ is a positive constant which depends only on n, φ, u and v.

If $p=2$ we have

$$\begin{aligned}
&\int_\Omega (h_n(u))^2\frac{|DT_K(\varphi(u)-\varphi(v))|^p}{K^2}dx\\
&\qquad\le\frac{\max\limits_{s\in[-2n,2n]}(\varphi'(s))}{K^2}\int_{U_K}\frac{1}{(\varphi'(u))}\times|D\varphi(u)-D\varphi(v)|^2dx.
\end{aligned}$$

As a consequence of (3.27) we have in both cases $1<p<2$ and $p=2$, for any $n>0$,

$$\lim_{K\to 0}\int_\Omega (h_n(u))^p\frac{|DT_K(\varphi(u)-\varphi(v))|^p}{K^p}dx=0. \tag{3.30}$$

In view of (3.28)–(3.30) and recalling that $h_n(u)$ converges to 1 almost everywhere in Ω and in $L^\infty(\Omega)$ weak-$*$ as n tends to infinity we conclude that

$$\int_\Omega \mathbb{1}_{\{\varphi(u)\neq\varphi(v)\}}dx=\lim_{n\to\infty}\lim_{K\to 0}\int_\Omega (h_n(u))^p\Big|\frac{T_K(\varphi(u)-\varphi(v))}{K}\Big|^p dx=0.$$

It follows that $\varphi(u)=\varphi(v)$ almost everywhere in Ω and since $\varphi'(s)\ge 1$ $\forall s\in\mathbb{R}$ we finally conclude that $u=v$ almost everywhere in Ω. □

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. We use standard real analysis to construct a function φ which verifies conditions (3.1)–(3.5) of Theorem 3.1. The proof will be divided into three steps.

Step 1. We claim that for any $\mu > 0$ there exists $\psi \in \mathscr{C}^1(\mathbb{R}^+)$ such that

$$\exists M > 0,\ \forall t \geq 0, \quad \begin{cases} 1 \leq \psi'(t) \leq M(\psi(t))^{1+\mu}, \\ 1 \leq \psi(t) \leq (\psi'(t))^{1+\mu}, \end{cases} \tag{3.31}$$

and $\forall s, t \in \mathbb{R}$

$$|a(x,s) - a(x,t)| \leq \Big| \int_s^t \psi(|z|)dz \Big|, \tag{3.32}$$

$$|\Phi(s) - \Phi(t)| \leq \Big| \int_s^t \psi(|z|)dz \Big|. \tag{3.33}$$

To this end it is sufficient to study, for any $n \geq 2$, the function ρ_n defined by

$$\forall t \geq 0, \quad \rho_n(t) = \Big[\int_0^t (1 + |w'(z)| + |w'(-z)|)dz + w(0) + 1 \Big]^n.$$

Due to (2.9) it is easy to check that $\forall t \geq 0$

$$1 \leq \rho_n'(t) \leq (C_1 + 1)n(\rho_n(t))^{1+\eta/n}, 1 \leq \rho_n(t) \leq \big(\rho_n'(t)\big)^{1+1/(n-1)}$$

and that inequalities (3.32) and (3.33) with ρ_n in place of ψ hold. Finally choosing n such that $\eta/n \leq \mu$ and $1/(n-1) \leq \mu$ leads to (3.31).

Step 2. Let $0 < \mu < 1$ and let $\psi \in \mathscr{C}^1(R^+)$ such that (3.31)–(3.33) hold true.

Let $\varphi(t) = \big((1 + \widetilde{\psi}(|t|))^3 - 1\big)\operatorname{sign}(t)$ $\forall t \in \mathbb{R}$ where we have set $\widetilde{\psi}(t) = \int_0^t \psi(z)dz$. We now derive some properties on φ and $\widetilde{\psi}$.

Due to the regularity of φ the function $\widetilde{\psi}$ belongs to $\mathscr{C}^1(\mathbb{R}) \cap \mathscr{C}^2(\mathbb{R}^*)$ and we have

$$\forall t \in \mathbb{R} \quad \varphi'(t) = 3\psi(|t|)(\widetilde{\psi}(|t|) + 1)^2, \tag{3.34}$$

$$\forall t \neq 0 \quad \varphi''(t) = 3(\widetilde{\psi}(|t|) + 1)(2\psi^2(|t|) + \psi'(|t|)(\widetilde{\psi}(|t|) + 1))\operatorname{sign}(t).$$

As ψ is an increasing function and due to (3.31) we have $\forall t > 0$

$$\begin{aligned} 1 \leq \varphi''(t) &\leq 6\big(\widetilde{\psi}(|t|) + 1\big)\psi(t)\Big(\int_0^t \psi'(z)dz + \psi(0)\Big) + 3\big(\psi(t)\big)^{1+\mu}M\big(\widetilde{\psi}(|t|) + 1\big)^2 \\ &\leq 6\big(\widetilde{\psi}(t) + 1\big)\psi(t)\Big(M\psi^{\mu}(t)\int_0^t \psi(z)dz + \psi(0)\Big) + 3M\big(\varphi'(t)\big)^{1+\mu} \\ &\leq 6\big(\psi(t)\big)^{1+\mu}M\big(\widetilde{\psi}(t) + 1\big)\big(\widetilde{\psi}(t) + \psi(0)\big) + 3M\big(\varphi'(t)\big)^{1+\mu} \\ &\leq M_1\big(\varphi'(t)\big)^{1+\mu}, \end{aligned} \tag{3.35}$$

where M_1 is a positive constant depending on M and $\psi(0)$. Moreover as far as the functions ψ and $\widetilde{\psi}$ are concerned inequalities (3.31) yield $\forall t > 0$

$$\psi(t)^{1-\mu} \leq M_2\big(\widetilde{\psi}(t) + 1\big), \tag{3.36}$$

$$\widetilde{\psi}(t) \leq M_3\big(\psi(t)\big)^{1+\mu+\mu^2}, \tag{3.37}$$

where M_2 and M_3 are positive constants.

Step 3. We claim that if μ is small enough then the function φ defined in Step 2 verifies (3.1)–(3.5) with δ depending of μ.

Let $0 < K \le K_0 < 1$ where K_0 will be chosen later and let $s, r \in \mathbb{R}$ such that $|\varphi(r) - \varphi(s)| \le K$. We now study the quantities

$$\Big|\frac{a(x,r)}{(\varphi'(r))^{p-1}} - \frac{a(x,s)}{(\varphi'(s))^{p-1}}\Big| \quad \text{and} \quad |\Phi(r) - \Phi(s)|$$

and we distinguish the cases $rs > 0$ and $rs \le 0$.

Case 1: we assume that $rs > 0$ and without loss of generality $0 < s < r$.

We first remark that if $\mu \le 1$ and if $M_1 K_0 < 1/2$ then (3.35) implies that

$$0 \le \varphi'(r) - \varphi'(s) \le \int_s^r \varphi''(z)dz \le M_1(\varphi'(r))^\mu \int_s^r \varphi'(z)dz \le M_1(\varphi'(r))K_0,$$

which gives

$$\frac{1}{2} \le \frac{\varphi'(r)}{\varphi'(s)} \le 1. \tag{3.38}$$

Let $K_0 > 0$ such that $M_1 K_0 < 1/2$ so that (3.5) holds for $rs > 0$.

We write

$$\begin{aligned}\Big|\frac{a(x,r)}{(\varphi'(r))^{p-1}} - \frac{a(x,s)}{(\varphi'(s))^{p-1}}\Big| &\le \frac{1}{(\varphi'(s))^{p-1}}|a(x,r) - a(x,s)| \\ &\quad + a(x,r)\Big|\frac{(\varphi'(r))^{p-1} - (\varphi'(s))^{p-1}}{(\varphi'(r))^{p-1}(\varphi'(s))^{p-1}}\Big|.\end{aligned} \tag{3.39}$$

For the first term in the right hand side of the above inequality using (3.32), the definition of φ and (3.34) we obtain

$$\begin{aligned}|a(x,r) - a(x,s)| &\le \int_s^r \psi(z)dz \le \frac{1}{3(\widetilde{\psi}(s)+1)^2}\int_s^r \varphi'(z)dz \\ &\le \frac{K}{3(1+\varphi(s))^{2/3}} \le \frac{M_4 K}{(1+\varphi(s)+\varphi(t))^{2/3}},\end{aligned}$$

where M_4 is a positive constant independent of r and s. It follows that

$$\frac{1}{(\varphi'(s))^{p-1}}|a(x,r) - a(x,s)| \le \frac{1}{(\varphi'(s))^{p-1}} \times \frac{M_4 K}{(1+\varphi(s)+\varphi(r))^{2/3}}. \tag{3.40}$$

The second term in the right hand side of (3.39) needs a few computations: using (3.32), (3.34)–(3.38) and standard real analysis arguments we have

$$\begin{aligned}a(x,r)&\Big|\frac{(\varphi'(r))^{p-1} - (\varphi'(s))^{p-1}}{(\varphi'(r))^{p-1}(\varphi'(s))^{p-1}}\Big| \\ &\le (p-1)\Big(\int_0^r \psi(z)dz + \|a(x,0)\|_{L^\infty(\Omega)}\Big)\frac{(\varphi'(s))^{p-2}\big(\varphi'(r) - \varphi'(s)\big)}{(\varphi'(r))^{p-1}(\varphi'(s))^{p-1}} \\ &\le M_5\big(1+\widetilde{\psi}(t)\big)\frac{\int_s^r \varphi''(z)dz}{(\varphi'(s))^{p-1}\varphi'(t)} \le \frac{M_5 M_2\big(1+\widetilde{\psi}(t)\big)K}{(\varphi'(s))^{p-1}(\varphi'(t))^{1-\mu}}.\end{aligned}$$

By (3.34) and (3.37) we deduce that

$$a(x,r)\Big|\frac{(\varphi'(r))^{p-1} - (\varphi'(s))^{p-1}}{(\varphi'(r))^{p-1}(\varphi'(s))^{p-1}}\Big| \le \frac{M_5 M_2 K}{(\varphi'(s))^{p-1}(1+\widetilde{\psi}(r))^{2-2\mu-\frac{2\mu+\mu^2}{1+\mu+\mu^2}}}.$$

Since $\varphi(s) \le \varphi(r)$ the definition of φ yields

$$a(x,r)\Big|\frac{(\varphi'(r))^{p-1}-(\varphi'(s))^{p-1}}{(\varphi'(r))^{p-1}(\varphi'(s))^{p-1}}\Big| \le \frac{M_6 K}{(\varphi'(s))^{p-1}(1+\varphi(r)+\varphi(s))^{\beta}}, \tag{3.41}$$

where

$$\beta = \frac{1}{3}\Big(2-2\mu-\frac{2\mu+\mu^2}{1+\mu+\mu^2}\Big).$$

We now turn to $|\Phi(r)-\Phi(s)|$. With similar arguments we get

$$\begin{aligned}|\Phi(r)-\Phi(s)| &\le \int_s^r \psi(z)dz \le \frac{1}{(1+\widetilde{\psi}(s))^2}\int_s^r \varphi'(z)dz\\ &\le \frac{3K}{(\varphi'(s))^{(p-1)/p}} \times \frac{(\psi(s))^{(p-1)/p}}{(1+\widetilde{\psi}(s))^{2/p}}\\ &\le \frac{3KM_3^{(p-1)(1-\mu)/p}}{(\varphi'(s))^{(p-1)/p}} \times \frac{1}{(1+\widetilde{\psi}(s))^{\frac{2}{p}-\frac{p-1}{p(1-\mu)}}}\\ &\le \frac{M_7 K}{(\varphi'(s))^{(p-1)/p}} \times \frac{1}{(1+\varphi(s)+\varphi(r))^{\theta}}\end{aligned} \tag{3.42}$$

where M_7 is a positive constant and where

$$\theta = \frac{1}{3p}\Big(2-\frac{p-1}{1-\mu}\Big). \tag{3.43}$$

We now define $\mu > 0$ and $\delta > 1/2$ such that the function φ verifies (3.2) and (3.3), (3.4) for s and r defined in this step. The definition of φ, (3.34) and (3.36) imply that

$$\frac{\varphi'}{(1+|\varphi|)^{1+\mu/(3(1-\mu))}} \in L^{\infty}(\mathbb{R}). \tag{3.44}$$

According to the definition of β it is easily seen that $\lim_{\mu\to 0}\beta = 2/3$. It follows that we can choose $\mu > 0$ (small enough) and define δ by

$$\delta = \frac{1}{3}\Big(2-2\mu-\frac{2\mu+\mu^2}{1+\mu+\mu^2}\Big)$$

such that

$$\frac{1}{2} < \delta < \frac{2}{3} \quad \text{and} \quad 1+\frac{\mu}{3(1-\mu)} < 2\delta. \tag{3.45}$$

Gathering (3.39), (3.40), (3.41) and (3.45) we deduce that

$$\Big|\frac{a(x,r)}{(\varphi'(r))^{p-1}} - \frac{a(x,s)}{(\varphi'(s))^{p-1}}\Big| \le \frac{M_8 K}{(\varphi'(s))^{p-1}(1+\varphi(r)+\varphi(s))^{\delta}},$$

that is (3.3) when $rs > 0$.

If $p = 2$ we observe that $\theta > 0$ and if $1 < p < 2$ we have $p\theta/(2-p) = 2/3 + (p-1)/(3(2-p)) > 2/3 > \delta$. Therefore (3.42) and (3.43) give that

$$|\Phi(r)-\Phi(s)| \le \frac{M_7 K}{(\varphi'(s))^{(p-1)/p}} \times \frac{1}{(1+\varphi(s)+\varphi(r))^{(2-p)\delta/p}},$$

that is (3.4) when $rs > 0$.

Case 2: we assume that $rs \le 0$.

Recalling that $\varphi' \geq 1$ and $|\varphi(r) - \varphi(s)| \leq K \leq K_0$ we have $|r| \leq K \leq K_0$ and $|s| \leq K \leq K_0$. Since the function φ belongs to $\mathscr{C}^1(\mathbb{R}) \cap \mathscr{C}^2(\mathbb{R}^*)$ it is easily seen that φ verifies (3.5) (with an appropriate value of L). By (3.32), (3.33), (3.39) and the regularity of φ we obtain that there exists $M_9 > 0$ (independent of r, s and K) such that

$$\Big| \frac{a(x,r)}{(\varphi'(r))^{p-1}} - \frac{a(x,s)}{(\varphi'(s))^{p-1}} \Big| \leq \frac{M_9 K}{(\varphi'(s))^{p-1}(1+|\varphi(r)|+|\varphi(s)|)^{\delta}},$$

$$|\Phi(r) - \Phi(s)| \leq \frac{M_9 K}{(\varphi'(s))^{(p-1)/p}} \times \frac{1}{(1+|\varphi(s)|+|\varphi(r)|)^{(2-p)\delta/p}}.$$

It follows that (3.4) and (3.5) hold for $rs \leq 0$.

We conclude that we have defined $K_0 > 0$, $\delta > 1/2$ such that in both cases $rs > 0$ and $rs \leq 0$ conditions (3.2), (3.3) and (3.4) are verified. In view of (3.44) and (3.45) we have also

$$\frac{\varphi'}{(1+|\varphi|)^{2\delta}} \in L^\infty(\mathbb{R}),$$

that is (2.8). The proof of Theorem 2.4 is complete. $\square$

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