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# WEAK-RENORMALIZED SOLUTION FOR A NONLINEAR BOUSSINESQ SYSTEM

Abdelatif Attaoui, Dominique Blanchard[1], and Olivier Guibé

Laboratoire de Mathématiques Raphaël Salem
Université de Rouen, CNRS
F-76801 Saint Etienne du Rouvray cedex, France

(Submitted by:Jesus Ildefonso Diaz)

**Abstract.** Abstract goes here We give a few existence results of a weak-renormalized solution for a class of nonlinear Boussinesq systems:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - 2\ \mathrm{div}\ (\mu(\theta)Du) + \nabla p = F(\theta) \qquad \text{in } \Omega\times(0,T),$$
$$\frac{\partial b(\theta)}{\partial t} + u\cdot\nabla b(\theta) - \Delta\theta = 2\mu(\theta)|Du|^2 \qquad \text{in } \Omega\times(0,T),$$
$$\mathrm{div}\ u = 0 \qquad \text{in } \Omega\times(0,T),$$

where $u$ is the velocity field of the fluid, $p$ is the pressure and $\theta$ is the temperature. The function $\mu(\theta)$ is the viscosity of the fluid and the function $F(\theta)$ is the buoyancy force which satisfies a growth assumption in dimension 2 and is bounded in dimension 3. Usual techniques for Navier-Stokes equations are mixed with the tools involved for renormalized solutions.

## 1. Introduction

In this paper, we deal with existence of a weak-renormalized solution for a class of nonlinear Boussinesq systems of the type:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - 2\ \mathrm{div}\ (\mu(\theta)Du) + \nabla p = F(\theta) \qquad \text{in } Q, \tag{1.1}$$
$$\frac{\partial b(\theta)}{\partial t} + u\cdot\nabla b(\theta) - \Delta\theta = 2\mu(\theta)|Du|^2 \qquad \text{in } Q, \tag{1.2}$$
$$\mathrm{div}\ u = 0 \qquad \text{in } Q, \tag{1.3}$$
$$u = 0 \text{ and } \theta = 0 \qquad \text{on } \Sigma_T, \tag{1.4}$$

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[1]Laboratoire Jacques-Louis Lions, Université Paris VI, Boîte courrier 187, 75252 Paris cedex 05

$$u(t=0)=u_0 \text{ and } b(\theta)(t=0)=b(\theta_0) \qquad \text{in } \Omega, \tag{1.5}$$

where $\Omega$ is an open, Lipschitz and bounded subset of $\mathbb{R}^N$ ($N=2$ or $N=3$), with boundary $\partial\Omega$, $T>0$, $Q=\Omega\times(0,T)$, $\Sigma_T=\partial\Omega\times(0,T)$. The unknowns are the displacement field $u:\Omega\times(0,T)\longrightarrow\mathbb{R}^N$ and the temperature field $\theta:\Omega\times(0,T)\longrightarrow\mathbb{R}$. The field $Du=\frac{1}{2}(\nabla u+(\nabla u)^t)$ is the so-called rate-deformation tensor field. Equation (1.1) is the conservation equation of momentum. In this equation, the quantities $\mu$ and $p$ respectively denote the kinematic viscosity and the pressure of the fluid so that the stress tensor in the incompressible fluid is given by the usual relation $\sigma=-pI_d+2\mu(\theta)Du$. The right hand side of equation (1.1) is the function $F(\theta)$, where $F$ is a force of gravity proportional to variations of density which depend on the temperature. The function $\mu$ is assumed to be continuous and bounded on $\mathbb{R}$. The function $F$ is continuous from $\mathbb{R}$ into $\mathbb{R}^N$, $u_0$ belongs to $(L^2(\Omega))^N$, with null divergence and $u_0\cdot n=0$ on $\partial\Omega$. Equation (1.2) is the energy conservation equation, in which the right hand side $\mu(\theta)|Du|^2$ is the dissipation energy. For this equation, the real valued function $b$ is assumed to be a strictly increasing $\mathcal{C}^1$-function defined on $\mathbb{R}$, $b(0)=0$ and $b'(r)\geq\alpha'\,\forall r\in\mathbb{R}$, for a constant $\alpha'>0$, the initial data $b(\theta_0)$ belongs to $L^1(\Omega)$. The Boussinesq system (1.1)–(1.5) of hydrodynamics equations (see [6]), arises from the coupling between a Navier-Stokes equation for the velocity and the pressure and an additional transport-diffusion equation for the temperature [19]. Systems which couple the Navier-Stokes equation with temperature diffusion are in particular studied in [12, 15, 16, 20, 21]. Nonlinear systems similar to (1.1)–(1.5) but with a constant right hand side (compared to $\theta$) and $b(\theta)=\theta$ have been in particular investigated in [7], [8] and [18]. In the particular case where the dissipation energy is neglected, existence and uniqueness result of a weak solution for system (1.1)–(1.5) (i.e. in the distribution meaning) has been established in [11]. Density gradients in a fluid are induced, for example, by temperature variations resulting from the non-uniform heating of the fluid. One will find, for example, a presentation of assumptions, which make it possible to justify the Boussinesq model in [1]. Let us emphasize that in simpler models the function $F$ is assumed to be linear (or even bounded) because of the linearization of the dependence of the density gradients with respect to the temperature. The model studied in this paper is more general than those which are described e.g in [1, 7, 8, 11, 18]. Indeed:
- the viscosity coefficient and the external forcing term are temperature-dependent (with nonlinear dependence).
- the internal energy is also assumed to be nonlinear with respect to the

temperature and this affects the time derivative term in the temperature equation.
- there is a right hand side in the energy conservation equation which is quadratic in the spatial gradient of the velocity field.
Existence of solutions of (1.1)–(1.5) is based on stability of equations (1.1) and (1.2) if approximation arguments are used, or on the uniqueness of solutions of these equations if one uses fixed-point arguments. We are thus constrained to distinguish the case of dimension 2 of space ($N = 2$) from dimension 3 ($N = 3$).

In the case of dimension 2, it is known that if $F(\theta) \in L^2(0,T;(H^{-1}(\Omega))^2)$, then the Navier-Stokes equation (1.1) has a unique solution for $u_0 \in (L^2(\Omega))^2$ and the dissipation energy $\mu(\theta)|Du|^2$ is stable in $L^1(Q)$ with respect to approximations. The energy conservation equation (1.2) is thus placed naturally within the $L^1$ framework. There are many works on parabolic equations with $L^1$ data (see e.g [3, 4, 9, 22]). To guarantee the uniqueness and the stability of the solution of (1.2), we use the framework of renormalized solutions which have these properties contrary to the weak solutions. This notion has been introduced by R.-J. DiPerna and P.-L. Lions in [13] and [14] for the study of Boltzmann equations (see also P-L. Lions [18] for applications to fluid mechanics models). This notion was then adapted to parabolic version for equations of type (1.2) with $L^1$ data (see e.g [2, 5]). The type of solutions which one obtains depends on the behavior of the function $F$. If, for example, $F$ is bounded, one obtains solutions for all given initial data $u_0 \in (L^2(\Omega))^2$ and $b(\theta_0) \in L^1(\Omega)$. To study the case of more general functions $F$, it is necessary to investigate the regularity of the solutions of (1.2). Under the assumptions that we adopt on $b$, the renormalized solutions of equation (1.2) satisfy the following regularities:

$$\theta \in L^\infty(0,T;L^1(\Omega)),$$

$$\forall\, k > 0, \int_0^T\!\!\int_\Omega |DT_k(\theta)|^2\, dx\, dt \le C\, k,$$

with $T_k(r)$ =min$(k,\ \max(r,-k))$ $\forall\, r \in \mathbb{R}$. We show then in a first step that $\theta \in L^r(0,T;L^q(\Omega))$ with $1 < q < \infty$ and $r < \frac{q}{q-1}$ (a similar result is shown in [23] for $N > 2$ but it cannot be used as such for $N = 2$). To have $F(\theta) \in L^2(0,T;(H^{-1}(\Omega))^2)$, we are constrained to make the following growth assumption on $F$:

$$\forall r \in \mathbb{R}, \qquad |F(r)| \le a + M|r|^\alpha,$$

with $a \geq 0$, $M \geq 0$ and $2\alpha \in [0,3[$. We then show in a second step that $F(\theta)$ is identified with an element of $L^2(0,T;(H^{-1}(\Omega))^2)$ with

$$\|F(\theta)\|_{L^2(0,T;(H^{-1}(\Omega))^2)} \leq C(a + \|\theta\|^{\alpha}_{L^r(0,T;L^q(\Omega))}).$$

These arguments allow us, thanks to approximations of $b$ and fixed-point methods, to show that (1.1)–(1.5) has solutions for small initial data.

In the case of dimension $N = 3$, the uniqueness of solution of the Navier-Stokes equation (1.1) and the stability of dissipation energy are open problems if $u_0$ belongs only to $(L^2(\Omega))^3$. If, for example, $u_0 \in (H^1_0(\Omega))^3$, and $F$ is bounded such that $\|u_0\|_{(H^1_0(\Omega))^3} + \|F\|_{(L^\infty(\mathbb{R}))^3} \leq \eta$, where $\eta$ is a constant small enough, we can then obtain the existence of a solution of (1.1)–(1.5) with the same techniques that in the case $N = 2$.

The paper is organized as follows: Section 2 is devoted to introduce the usual Navier-Stokes functional setting (according to the variational formulation introduced by Leray [17] within framework of free divergence functional spaces), to specify the assumptions on $b, F, \mu, u_0, \theta_0$ and $b(\theta_0)$ needed in the present study and to the definition of a weak-renormalized solution of (1.1)–(1.5). In Section 3, we describe the method used to prove existence of a solution through a fixed-point argument with respect to the unknown $\theta$. In Section 4, we investigate the existence, uniqueness and stability of the solution of the parabolic problem (3.5)–(3.7) resulting from (1.1)–(1.5). We assume in this section that $u$ is given in $L^2(0,T;(H^1_0(\Omega))^2) \cap L^\infty(0,T;(L^2(\Omega))^2)$ with div $u = 0$ and we will mainly used the results of [5]. In Section 5, we deal with the existence of a solution of (1.1)–(1.5) for $N = 2$. We distinguish four cases according to the values of $\alpha$. For $\alpha = 0$ ($F$ is bounded), we introduce an approximate problem of the system (1.1)–(1.5) by regularizing the function $b$. We prove that this problem admits a weak-renormalized solution for all initial data by using the Schauder's fixed-point theorem. The existence of a weak-renormalized solution of the coupled system is then obtained by passing to the limit in this approximate problem. For $0 < 2\alpha \leq 1$, we introduce an approximate problem of the system (1.1)–(1.5) by regularizing of the function $F$ by $F^\varepsilon$ ($F^\varepsilon$ being continuous and bounded). Then, we can use the result of the first case ($\alpha = 0$) to deduce that there exists a weak-renormalized solution of this approximate problem for all initial data and we will pass to the limit in this problem to obtain the existence of a solution of (1.1)–(1.5). For the last cases where $1 < 2\alpha < 2$ and $2 \leq 2\alpha < 3$, we introduce an approximate problem of the system (1.1)–(1.5) by regularizing of the function $b$. For small initial data, the Schauder's fixed-point theorem ensures the existence of a weak-renormalized solution of this problem and we pass to the limit as in the preceeding sections. In Section 6, we deal with dimension $N = 3$. In the particular case, where $F$ is bounded in $L^\infty$ and $u_0 \in (H_0^1(\Omega))^3$, we prove the existence of a weak-renormalized solution of the coupled system for small data $F$ and $u_0$.

## 2. Assumptions and definition of a weak-renormalized solution

Throughout the paper, we assume that the following assumptions hold true: $\Omega$ is an open, Lipschitz and bounded subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) with boundary $\partial\Omega$, $T > 0$ is given and we set $Q = \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$.
We introduce the usual Navier-Stokes functional setting:

$$
\begin{aligned}
C_\sigma^\infty(\Omega) &= \{u \in C_0^\infty(\Omega; \mathbb{R}^N); \text{div } u = 0\},\\
L_\sigma^p(\Omega) &= \text{closure of } C_\sigma^\infty(\Omega) \text{ in } L^p(\Omega; \mathbb{R}^N),\\
H_\sigma^1(\Omega) &= \text{closure of } C_\sigma^\infty(\Omega) \text{ in } H_0^1(\Omega; \mathbb{R}^N),\\
L_\sigma^p(Q) &= L^p(0, T; L_\sigma^p(\Omega)),
\end{aligned}
$$

when $p \geq 1$. We assume that the following assumptions hold true:

$$b \text{ is a strictly increasing } C^1\text{-function defined on } \mathbb{R} \text{ such that } b(0) = 0, \tag{2.1}$$

$$b'(r) \geq \alpha' \, \forall r \in \mathbb{R} \text{ for a constant } \alpha' > 0, \tag{2.2}$$

$$\mu \text{ is continuous on } \mathbb{R}, \text{ such that } m_0 \leq \mu(s) \leq m_1, \forall s \in \mathbb{R} \text{ with } 0 < m_0 \leq m_1, \tag{2.3}$$

$$F \text{ is continuous and satisfies the growth assumption:} \quad \forall r \in \mathbb{R} \quad |F(r)| \leq a + M|r|^\alpha \text{ with } a \geq 0, M \geq 0 \text{ and } 0 \leq 2\alpha < 3, \tag{2.4}$$

$$u_0 \in (L^2(\Omega))^N, \text{ div } u_0 = 0 \text{ and } u_0 \cdot n = 0 \text{ on } \partial\Omega, \tag{2.5}$$

$$\theta_0 \text{ is a measurable function defined on } \Omega \text{ such that } b(\theta_0) \in L^1(\Omega). \tag{2.6}$$

In dimension $N = 3$ (Section 6), we adopt stronger assumptions than (2.4) and (2.5) i.e. $F$ is bounded in $L^\infty$ ($\alpha = 0$) and $u_0 \in (H_0^1(\Omega))^3$.

As usual, the pressure $p$ is eliminated from the system (1.1)–(1.5). The De Rham's lemma [10] allows to recover this unknown. In the sequel we study the following system:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - 2 \text{ div } (\mu(\theta) Du) = F(\theta) \quad \text{in } (H_\sigma^1)'(\Omega), \quad \text{for almost every } t \in (0, T), \tag{2.7}$$

$$\frac{\partial b(\theta)}{\partial t} + u\cdot\nabla b(\theta) - \Delta\theta = 2\mu(\theta)|Du|^2 \qquad \text{in } Q, \tag{2.8}$$

$$\operatorname{div} u = 0 \qquad \text{in } Q, \tag{2.9}$$

$$u = 0 \text{ and } \theta = 0 \qquad \text{on } \Sigma_T, \tag{2.10}$$

$$u(t=0) = u_0 \text{ and } b(\theta)(t=0) = b(\theta_0) \qquad \text{in } \Omega. \tag{2.11}$$

For any measurable function $\theta$ defined on $Q$, we consider the bilinear and trilinear forms usually used in the weak formulation of the Navier-Stokes equations:

$$a_\theta(u,v) = \frac{1}{2}\sum_{i,j=1}^{N}\int_\Omega \mu(\theta)(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\frac{\partial v_j}{\partial x_i}\,dx,$$

$$d(u,v,w) = \sum_{i,j=1}^{N}\int_\Omega u_j\frac{\partial v_i}{\partial x_j}w_j\,dx = \int_\Omega (u\cdot\nabla)v\cdot w\,dx,$$

$\forall u, v \in H^1_\sigma(\Omega), \forall w \in H^1_\sigma(\Omega)\cap L^N_\sigma(\Omega)$.

We recall that $a_\theta$ is continuous and coercive in $H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)$ for a.e. $t\in[0,T]$ and that $d$ is anti-symmetric and continuous in $H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)\times(H^1_\sigma(\Omega)\cap L^N_\sigma(\Omega))$.

**Definition 2.1.** *A couple of functions $(\theta, u)$ defined on $\Omega\times(0,T)$ is called a weak-renormalized solution of problem (2.7)–(2.11) if $u$ and $\theta$ satisfy:*

$$u \in L^2(0,T;H^1_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega)), \tag{2.12}$$

$$T_K(\theta) \in L^2(0,T;H^1_0(\Omega)) \textit{ for any } K\ge 0 \textit{ and } b(\theta)\in L^\infty(0,T;L^1(\Omega)), \tag{2.13}$$

$$\int_{\{(x,t)\in Q; n\le|b(\theta)(x,t)|\le n+1\}} b'(\theta)|D\theta|^2\,dx\,dt \longrightarrow 0 \textit{ as } n\to+\infty, \tag{2.14}$$

$$\langle u_t, w\rangle_{L^2_\sigma(\Omega)} + a_\theta(u,w) + d(u,u,w) = \langle F(\theta), w\rangle \quad \forall w\in H^1_\sigma(\Omega)\cap L^N_\sigma(\Omega), \tag{2.15}$$

$$u(t=0) = u_0 \textit{ a.e in } \Omega, \tag{2.16}$$

*$\forall\, S\in C^\infty(\mathbb{R})$ such that $S'$ has a compact support, we have*

$$\frac{\partial S(b(\theta))}{\partial t} + \operatorname{div}(uS(b(\theta))) - \operatorname{div}(S'(b(\theta))D\theta) + S''(b(\theta))b'(\theta)|D\theta|^2 = 2\mu(\theta)|Du|^2 S'(b(\theta)) \textit{ in } \mathcal{D}'(Q), \tag{2.17}$$

$$S(b(\theta))(t=0) = S(b(\theta_0)) \textit{ in } \Omega. \tag{2.18}$$

## 3. The fixed-point argument

In this section, we describe the (standard) method used to prove existence of a solution through a fixed-point argument with respect to the unknown $\theta$. Let us notice that it requires an additional assumption on the function $b$ (at least if one uses standard methods developed e.g in [5], see section 4).

Let $L$ be a Lebesgue's space of the type $L = L^r(0,T;L^q(\Omega))$ $(r,q \geq 1)$. For a fixed $\theta \in L$, let us consider the Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - 2\ \mathrm{div}\ (\mu(\theta)Du) = F(\theta) \quad \text{in } (H^1_\sigma)'(\Omega), \tag{3.1}$$

for almost every $t \in (0,T)$,

$$\mathrm{div}\ u = 0 \quad \text{in } Q, \tag{3.2}$$

$$u = 0 \quad \text{on } \Sigma_T, \tag{3.3}$$

$$u(t=0) = u_0 \quad \text{in } \Omega. \tag{3.4}$$

Suppose that (3.1)–(3.4) admit a unique solution $u \in L^2(0,T;H^1_\sigma(\Omega))$ so that $\mu(\theta)|Du|^2 \in L^1(Q)$. Indeed, this is the case if $F(\theta) \in L^2(0,T;(H^{-1}(\Omega))^N)$. Then, we consider the parabolic problem:

$$\frac{\partial b(\hat\theta)}{\partial t} + u\cdot\nabla b(\hat\theta) - \Delta\hat\theta = 2\mu(\theta)|Du|^2 \quad \text{in } Q, \tag{3.5}$$

$$\hat\theta = 0 \quad \text{on } \Sigma_T, \tag{3.6}$$

$$b(\hat\theta)(t=0) = b(\theta_0) \quad \text{in } \Omega. \tag{3.7}$$

Assume that the hypotheses on the data insure that (3.5)–(3.7) admit a unique renormalized solution $\hat\theta$. In order to apply a fixed-point argument, it is first necessary to have $\hat\theta \in L$ so that we can consider the mapping

$$\psi : \theta \longrightarrow \hat\theta$$

from $L$ into $L$.

As a consequence, the value of $\alpha$ must be such that the regularity of the renormalized solution of (3.5)–(3.7) implies $F(\theta) \in L^2(0,T;(H^{-1}(\Omega))^N)$. This leads to different choices of $L$ depending of the range of $\alpha$. Secondly, we use the stability of renormalized solution with respect to the data and the stability of the quantity $\mu(\theta)|Du|^2$ (with respect to approximation processes) to show that $\psi$ is continuous and compact. At last, in order to show that there exists a ball $B$ of $L$ such that $\psi(B) \subset B$, we distinguish two cases: if $0 \leq 2\alpha \leq 1$, this is proved for any data satisfying (2.5)–(2.6), while if

$1 < 2\alpha < 3$, we are led to assume that $a$, $\|b(\theta_0)\|_{L^1(\Omega)}$ and $\|u_0\|_{(L^2(\Omega))^N}$ are small enough.

## 4. The parabolic problem

In this section, we investigate the existence, uniqueness and stability of the solution of (3.5)–(3.7). There are now a large number of papers on the properties of renormalized (or entropy) solutions for this type of problems ([3], [4], [5], [9], [18], [22], [23]) and we will mainly used the results of [5]. We assume in this section that $u$ is given in $L^2(0,T;H^1_\sigma(\Omega))\cap L^\infty(0,T;L^2_\sigma(\Omega))$ with div $u=0$. Moreover the function $\theta$ is given in a Lebesgue space $L$ so that assumption (2.4) implies that $f=\mu(\theta)|Du|^2\in L^1(Q)$. We prove the following two lemmas (most of the results being standard).

**Lemma 4.1.** *Under the assumptions (2.1), (2.2), (2.3) and (2.6), the problem (3.5)–(3.7) admits at least a renormalized solution. If $b'$ is locally Lipschitz-continuous, the solution of (3.5)–(3.7) is unique. Let $b_\varepsilon$ be a sequence of $\mathcal{C}^2$-approximations of $b$ such that $b'_\varepsilon(r)>0, \forall r\in\mathbb{R}$, $b_\varepsilon(0)=0$, and such that $b_\varepsilon$ and $b'_\varepsilon$ converge to $b$ and $b'$ uniformly on $\mathbb{R}$ as $\varepsilon$ tends to $0$. Let $f^\varepsilon$ be a sequence of $L^1(Q)$. Let us denote by $\hat\theta^\varepsilon$ the unique renormalized solution of (3.5)–(3.7) with $b_\varepsilon$ and $f^\varepsilon$ in place of $b$ and $2\mu(\theta)|Du|^2$. Then :*
*· if $f^\varepsilon$ is bounded in $L^1(Q)$, then there exists a subsequence of $\hat\theta^\varepsilon$ such that*

$$\hat\theta^\varepsilon \longrightarrow v \ \text{a.e. in } Q, \tag{4.1}$$

$$T_K(\hat\theta^\varepsilon) \rightharpoonup T_K(v) \ \text{weakly in } L^2(0,T;H^1_0(\Omega)), \tag{4.2}$$

*as $\varepsilon$ tends to $0$, for any $K>0$ , where $v$ is a measurable function defined on $Q$.*
*· if $f^\varepsilon$ strongly converges to $2\mu(\theta)|Du|^2$ in $L^1(Q)$, then there exists a subsequence of $\hat\theta^\varepsilon$ such that*

$$\hat\theta^\varepsilon \longrightarrow \hat\theta \ \text{a.e. in } Q, \tag{4.3}$$

$$T_K(\hat\theta^\varepsilon) \to T_K(\hat\theta) \ \text{strongly in } L^2(0,T;H^1_0(\Omega)), \tag{4.4}$$

*as $\varepsilon$ tends to $0$, for any $K>0$, and $\hat\theta$ is a renormalized solution of (3.5)–(3.7).*

The following lemma gives a regularity result of renormalized solution of (3.5)–(3.7) for dimension $N\ge 1$.

**Lemma 4.2.** *Under the assumptions (2.1), (2.2), (2.3) and (2.6), any renormalized solution $\hat\theta$ of (3.5)–(3.7) satisfies the following estimates:*

*- for $N \geq 1$ and all $p \in [1, \frac{N+2}{N}[$, there exists a constant $C$ (depending only on $p$, $N$, $\Omega$, and $T$) such that:*

$$\|\hat{\theta}\|_{L^p(Q)} \leq C\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)}).$$

*- for $N = 2$, for all $q, r$ such that $1 < q < \infty$, and $1 \leq r < \frac{q}{q-1}$, we have $\hat{\theta} \in L^r(0,T; L^q(\Omega))$, and there exists a constant $C$ (depending on $T, r, q, \Omega$) such that*

$$\|\hat{\theta}\|_{L^r(0,T;L^q(\Omega))} \leq C\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)}).$$

*Proof of Lemma 4.1.* The proof is almost identical of the one given in, e.g [5] where the result is established for $u \equiv 0$ and we just sketch the arguments involving the term $u \cdot Db(\hat{\theta})$. Loosely speaking, this term does not affect the estimates on $b(\hat{\theta})$ and $\hat{\theta}$ since its contribution against test functions of the type $\phi(\hat{\theta})$ is equal to zero because div $u = 0$ and of the boundary conditions (2.10). Indeed, the proof of Lemma 4.1 is performed through approximation and passage to the limit. The functions $\mu(\theta)|Du|^2$ and $\theta_0$ are approximated by smooth functions. The function $b$ is suppose to be Lipschitz on $\mathbb{R}$ and, as in [11], the function $u$ is approximated in $L^2(Q)$ by a sequence $u_j \in L^\infty(Q) \cap L^2_\sigma(Q)$ (then div $u_j = 0$ in $Q$). The corresponding problem indeed admits a weak solution $\theta_j \in L^2(0,T; H^1_0(\Omega))$ with $b(\theta_j) \in L^\infty(0,T; L^2(\Omega))$. To pass to the limit in the term $u_j \cdot Db(\theta_j)$ with respect to $j$ is easy because (by standard argument) $b(\theta_j) \rightharpoonup b(\theta)$ weakly in $L^2(Q)$ (recall that $b$ is also supposed to be Lipschitz-continuous on $\mathbb{R}$), and $u_j \longrightarrow u$ strongly in $L^2(Q)$. It follows that the approximate problem with respect of $b$, $\theta_0$ and $\mu(\theta)|Du|^2$ admits at least a weak solution $\hat{\theta}^\varepsilon \in L^2(0,T; H^1_0(\Omega))$ with $b(\hat{\theta}^\varepsilon) \in L^2(0,T; H^1_0(\Omega))$. As mentioned above, we can repeat exactly the same procedure as in [5] to show that (for a subsequence):

$$\hat{\theta}^\varepsilon \longrightarrow \hat{\theta} \text{ a.e. in } Q, \tag{4.5}$$

$$T_K(\hat{\theta}^\varepsilon) \rightarrow T_K(\hat{\theta}) \text{ strongly in } L^2(0,T; H^1_0(\Omega)), \tag{4.6}$$

as $\varepsilon$ tends to zero, for any $K > 0$, because the convection term $u \cdot Db(\theta)$ never contributes in all the derivations of [5] (see Lemma 1 and Theorem 1 of that paper). As a consequence, all we have to show here is firstly that the "renormalized term" $u \cdot DS(b(\hat{\theta}^\varepsilon))$ passes to the limit as $\varepsilon$ tends to 0 for any function $S \in \mathcal{C}^\infty(\mathbb{R})$ such that $S'$ has a compact support and secondly that the initial condition $S(b(\hat{\theta}))(t = 0) = S(b(\theta_0))$ holds true. Indeed, we have $u \cdot DS(b(\hat{\theta}^\varepsilon)) = u \cdot S'(b(T_k(\hat{\theta}^\varepsilon)))b'(T_k(\hat{\theta}^\varepsilon))DT_k(\hat{\theta}^\varepsilon)$ for some $k$ since $S'$ has a compact support and $b'(r) \geq \alpha' \,\forall r \in \mathbb{R}$ (see (2.2)). Due to (4.5) and (4.6), the sequence $u \cdot DS(b(\hat{\theta}^\varepsilon))$ strongly converges in $L^1(Q)$ to $u \cdot DS(b(\hat{\theta}))$. To recover the initial condition (3.7), we proceed again as in [5] upon remarking that the term $u \cdot DS(b(\hat{\theta}^\varepsilon))$ is compact in $L^1(Q)$.

The stability and uniqueness results can be proved exactly as in [5]. □

*Proof of Lemma 4.2.* Any renormalized solution $\hat{\theta}$ of (3.5)–(3.7) satisfies the usual estimates (see e.g. [2] and [5]))

$$\int_Q |DT_K(\hat{\theta})|^2 dxdt \le K(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)}), \tag{4.7}$$

and

$$\|b(\hat{\theta})\|_{L^\infty(0,T;L^1(\Omega))} \le \|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)}. \tag{4.8}$$

Estimate (4.7) and Lemma 1 of [2] gives that for any $p \in [1, \frac{N+2}{N}[$, there exists a constant $C$ (depending only on $p$, $N$, $\Omega$, and $T$ ) such that:

$$\|\hat{\theta}\|_{L^p(Q)} \le C\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)})^{\frac{N}{N+2}} \|\hat{\theta}\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}. \tag{4.9}$$

Now, assumption (2.2) and estimate (4.8) give:

$$\|\hat{\theta}\|_{L^\infty(0,T;L^1(\Omega))} \le \frac{1}{\alpha}\,(\|\mu(\theta)|Du|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)}), \tag{4.10}$$

The first part of the Lemma 4.2 follows directly from (4.9) and (4.10).

Now, we turn to the proof of the second part of Lemma 4.2. A similar result was shown in [23] in the case where $N > 2$.

Since $\hat{\theta}$ is a renormalized solution of (3.5)–(3.7) and in view of (4.7) and (4.10), we have:

$$\|\hat{\theta}\|_{L^\infty(0,T;L^1(\Omega))} \le M, \tag{4.11}$$

$$\forall K > 0\ ,\ \int_Q |DT_K(\hat{\theta})|^2\,dx\,dt \le K\,M. \tag{4.12}$$

Our goal is to show that there exists a constant $C$ independent on $M$ such that:

$$\|\hat{\theta}\|_{L^r(0,T;L^q(\Omega))} \le C\,M.$$

By Gagliardo-Nirenberg's inequality and (4.11), we have:

$$\begin{aligned}\int_\Omega |T_K(\hat{\theta})|^3\,dx &\le C\int_\Omega |T_K(\hat{\theta})|\,dx \int_\Omega |DT_K(\hat{\theta})|^2\,dx,\\ &\le C\,M\int_\Omega |DT_K(\hat{\theta})|^2\,dx,\end{aligned}$$

for almost any $t$ in $(0,T)$, where $C$ is a constant depending on $\Omega$.
This gives on the one hand:

$$\text{meas}\{(x,t) : |\hat{\theta}| > K\} \leq C\,\frac{M}{K^3}\int_\Omega |DT_K(\hat{\theta})|^2\,dx, \tag{4.13}$$

for almost any $t$ in $(0,T)$.
On the other hand, estimate (4.11) leads to:

$$\text{meas}\{x \in \Omega : |\hat{\theta}| > K\} \leq \frac{M}{K}, \tag{4.14}$$

for almost any $t$ in $(0,T)$.
Now for any $0 \leq \sigma \leq 1$ we have for almost any $t$ in $(0,T)$

$$\text{meas}\{x \in \Omega : |\hat{\theta}(x,t)|^q > s\} =$$
$$\left(\text{meas}\{x \in \Omega : |\hat{\theta}(x,t)| > s^{\frac{1}{q}}\}\right)^{\sigma}\left(\text{meas}\{x \in \Omega : |\hat{\theta}(x,t)| > s^{\frac{1}{q}}\}\right)^{1-\sigma}.$$

In view of (4.13) and (4.14), we obtain:

$$\text{meas}\{x \in \Omega : |\hat{\theta}(x,t)|^q > s\} \leq CM^{\sigma}\left(\frac{\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat{\theta})|^2\,dx}{s^{\frac{3}{q}}}\right)^{\sigma}\frac{M^{1-\sigma}}{s^{\frac{1-\sigma}{q}}},$$

from which we deduce that:

$$\text{meas}\{x \in \Omega : |\hat{\theta}(x,t)|^q > s\} \leq CM\left(\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat{\theta})|^2\,dx\right)^{\sigma}\frac{1}{s^{\frac{1+2\sigma}{q}}}, \tag{4.15}$$

for almost $t$ in $(0,T)$.
In the sequel, the proof of the lemma will be divided into three steps.
**Step 1: $q = r$.** For any real number $1 < q$, we write:

$$\int_0^T\left(\int_\Omega |\hat{\theta}|^q\,dx\right)dt = \int_0^T\left(\int_0^\infty \text{meas}\{x \in \Omega : |\hat{\theta}|^q > s\}\,ds\right)dt,$$

so that for any real number $\beta > 0$

$$\int_0^T\int_\Omega |\hat{\theta}|^q dx\,dt \leq \int_0^T\int_0^\beta \text{meas}\{x \in \Omega : |\hat{\theta}|^q > s\}\,ds\,dt$$
$$+\int_0^T\int_\beta^\infty \text{meas}\{x \in \Omega : |\hat{\theta}|^q > s\}\,ds\,dt.$$

Due to (4.15), we obtain:

$$\int_0^T\int_\Omega |\hat{\theta}|^q dx\,dt \leq \beta T|\Omega| + CM\int_0^T\int_\beta^\infty\int_\Omega |DT_{s^{\frac{1}{q}}}(\hat{\theta})|^2\,dx\frac{1}{s^{\frac{3}{q}}}\,ds\,dt.$$

Fubini's theorem and (4.12) then give

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le \beta T|\Omega| + CM\int_\beta^\infty\int_0^T\int_\Omega |DT_{s^{\frac1q}}(\hat\theta)|^2\,dx\frac{1}{s^{\frac3q}}\,dt\,ds,$$
$$\le \beta T|\Omega| + CM^2\int_\beta^\infty \frac{1}{s^{\frac2q}}\,ds.$$

Since $q<2$ because $q=r<\frac{q}{q-1}$, then:

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le \beta T|\Omega| + CM^2\frac{\beta^{1-\frac2q}}{\frac2q-1}.$$

Choosing $\beta = M^q$ in the above inequality finally gives:

$$\int_0^T\int_\Omega |\hat\theta|^q dx\,dt \le M^q(T|\Omega| + \frac{Cq}{2-q}),$$

which establishes the second part of Lemma 4.2 in the case where $r=q$.

**Step 2:** $q<r$**.** For all real numbers $1<q$ and $r>q$, we write:

$$\int_0^T\left(\int_\Omega|\hat\theta|^q\,dx\right)^{\frac rq}dt = \int_0^T\left(\int_0^\infty \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\right)^{\frac rq}dt,$$

so that for any positive real number $\beta$, we have:

$$\int_0^T\left(\int_\Omega|\hat\theta|^q\,dx\right)^{\frac rq}dt \le \int_0^T\left(\int_0^\beta \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\right)^{\frac rq}dt$$
$$+\int_0^T\left(\int_\beta^\infty \text{meas}\{x\in\Omega : |\hat\theta|^q > s\}\,ds\right)^{\frac rq}dt.$$

Using (4.15) with $\sigma=\frac qr$ in the above inequality gives

$$\int_0^T\left(\int_\Omega|\hat\theta|^q\,dx\right)^{\frac rq}dt \le \beta^{\frac rq}|\Omega|^{\frac rq}T$$
$$+C\int_0^T\left(\int_\beta^\infty \frac{M\left(\int_\Omega|DT_{s^{\frac1q}}(\hat\theta)|^2\,dx\right)^{\frac qr}}{s^{\frac1q+\frac2r}}\,ds\right)^{\frac rq}dt.$$

Writing for a real number $\gamma>1$

$$\left(\int_\Omega|DT_{s^{\frac1q}}(\hat\theta)|^2\,dx\right)^{\frac qr} = \left(\frac{\int_\Omega|DT_{s^{\frac1q}}(\hat\theta)|^2\,dx}{s^{\frac1q+\gamma}}\right)^{\frac qr}s^{\frac1r+\frac{\gamma q}{r}},$$

for almost $t$ in $(0,T)$ and using Hölder's inequality lead to

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}\,|\Omega|^{\frac{r}{q}}\,T$$

$$+CM^{\frac{r}{q}}\left[\int_\beta^\infty \frac{ds}{s^{[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q}}}\right]^{\frac{r-q}{q}}\int_0^T\left[\int_\beta^\infty \frac{\int_\Omega |DT_{s^{\frac1q}}(\hat\theta)|^2\,dx}{s^{\frac1q+\gamma}}\right]ds\,dt.$$

Notice that in order to have $\left[\int_\beta^\infty \frac{ds}{s^{[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q}}}\right]^{\frac{r-q}{q}} < +\infty$, we must have $[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q} > 1$. With the help of Fubini's theorem and (4.12), the above inequality gives

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}|\Omega|^{\frac{r}{q}}T$$

$$+\,CM^{1+\frac{r}{q}}\int_\beta^\infty \frac{1}{s^\gamma}ds\cdot\left[\int_\beta^\infty \frac{ds}{s^{[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q}}}\right]^{\frac{r-q}{q}}.$$

Since $r<\frac{q}{q-1}$ (by hypothesis of Lemma 4.2), then $1+\frac1q+\frac{r}{q^2}-\frac{r}{q}>1$ and there exists a real number $\gamma$ such that $1<\gamma<1+\frac1q+\frac{r}{q^2}-\frac{r}{q}$ which in turn insures that $[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q}>1$.
For such a choice of $\gamma$ it follows that

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le \beta^{\frac{r}{q}}|\Omega|^{\frac{r}{q}}T + CM^{1+\frac{r}{q}}\,\frac{\beta^{1-\gamma}}{\gamma-1}\cdot\frac{\beta^{\left[1-[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q}\right]\frac{r-q}{q}}}{[\frac1q+\frac1r(1-q\gamma)]\frac{r}{r-q}-1}.$$

Choosing now $\beta=M^q$, we conclude that

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le C(\gamma,r,q,N)M^r,$$

which proves the second part of Lemma 4.2 in the case where $q<r$.

**Step 3:** $r<q$**.** If $q<2$ then $r<2$ and by Hölder's inequality and the analysis of the first case, we obtain:

$$\int_0^T\Big(\int_\Omega |\hat\theta|^q\,dx\Big)^{\frac{r}{q}}dt \le C(r,q,\Omega,T)M^r.$$

If $q \geq 2$, Hölder's inequality implies that for $0 < \sigma < 1$

$$\Big( \int_\Omega |\hat\theta|^q \, dx \Big)^{\frac{r}{q}} \leq \Big( \int_\Omega |\hat\theta| \, dx \Big)^{\frac{\sigma\, r}{q}} \Big( \int_\Omega |\hat\theta|^{\frac{q-\sigma}{1-\sigma}} \, dx \Big)^{\frac{(1-\sigma)r}{q}},$$

for almost $t$ in $(0,T)$, which gives using (4.11)

$$\int_0^T \Big( \int_\Omega |\hat\theta|^q \, dx \Big)^{\frac{r}{q}} dt \leq M^{\frac{\sigma\, r}{q}} \int_0^T \Big( \int_\Omega |\hat\theta|^{\frac{q-\sigma}{1-\sigma}} \, dx \Big)^{\frac{1-\sigma}{q-\sigma} \frac{r(q-\sigma)}{q}} dt. \tag{4.16}$$

In what follows, we show that we can choose $0 < \sigma < 1$ such that

$$\int_0^T \Big( \int_\Omega |\hat\theta|^{\frac{q-\sigma}{1-\sigma}} \, dx \Big)^{\frac{1-\sigma}{q-\sigma} \frac{r(q-\sigma)}{q}} dt \leq CM. \tag{4.17}$$

To this end, recall that $N = 2$, so that Sobolev's embedding theorem gives

$$\int_\Omega |T_K(\hat\theta)|^p \, dx \leq C(p, |\Omega|) \Big( \int_\Omega |DT_K(\hat\theta)|^2 \, dx \Big)^{\frac{p}{2}}, \tag{4.18}$$

for any $p \geq 2$.
Since for $p = \frac{2q}{(1-\sigma)r}$

$$|T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}} \leq |T_K(\hat\theta)|^p \cdot K^{\frac{1}{1-\sigma}(q-\sigma-\frac{2q}{r})},$$

we have

$$\Big( \int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}} \, dx \Big)^{\frac{1-\sigma}{q-\sigma} \frac{r(q-\sigma)}{q}} \leq K^{\frac{r}{q}(q-\sigma-\frac{2q}{r})} \Big( \int_\Omega |T_K(\hat\theta)|^p \, dx \Big)^{\frac{2}{p}},$$

for almost any $t$ in $(0,T)$.
In view of (4.18), we deduce that:

$$\Big( \int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}} \, dx \Big)^{\frac{1-\sigma}{q-\sigma} \frac{r(q-\sigma)}{q}} \leq c\, K^{\frac{r}{q}(q-\sigma-\frac{2q}{r})} \int_\Omega |DT_K(\hat\theta)|^2 \, dx,$$

for almost $t$ in $(0,T)$.
With (4.12), it implies that for any $K > 0$

$$\int_0^T \Big( \int_\Omega |T_K(\hat\theta)|^{\frac{q-\sigma}{1-\sigma}} \, dx \Big)^{\frac{(1-\sigma)r}{q}} dt \leq CK^{\frac{r}{q}(q-\sigma-\frac{2q}{r})} K\, M.$$

Since $r < \frac{q}{q-1}$, we can take $\sigma = \frac{q}{r}(r-1)$ and Fatou's lemma implies that (4.17) holds true. Inserting (4.17) into (4.16) finally yields

$$\int_0^T \left( \int_\Omega |\hat{\theta}|^q \, dx \right)^{\frac{r}{q}} dt \leq C(\gamma, r, q, N) M^r.$$

This achieves the proof of Lemma 4.2. □

## 5. Existence of a solution for N=2

This section is devoted to establish the following existence theorem:

**Theorem 5.1.** *Assume that the assumptions (2.1)–(2.6) on the data hold true. Then:*
*- if $0 \leq 2\alpha \leq 1$, there exists at least a weak-renormalized solution of problem (2.7)–(2.11) (in the sense of Definition 2.1).*
*- if $1 < 2\alpha < 3$, there exists a real positive number $\eta$ such that if $a + \|u_0\|_{(L^2(\Omega))^2} + \|b(\theta_0)\|_{L^1(\Omega)} \leq \eta$, there exists at least a weak-renormalized solution of problem (2.7)–(2.11) (in the sense of Definition 2.1).*

*Proof of Theorem 5.1.* We use the fixed point-argument described in Section 3 and we distinguish four cases according to the values of $\alpha$.

**CASE 1: $\alpha = 0$.**

For a fixed $\theta \in L^1(Q)$, since $F$ is bounded ($\alpha = 0$), we denote by $u$ the unique weak solution of (3.1)–(3.4) in $L^2(0,T; H^1_\sigma(\Omega)) \cap L^\infty(0,T; L^2_\sigma(\Omega))$ (see e.g [18] and [26]). As in Section 4, $b_\varepsilon$ is a sequence of $\mathcal{C}^2$-approximations of $b$ such that $b'_\varepsilon$ is a locally Lipschitz-continuous on $\mathbb{R}$ and $b'_\varepsilon$ converges to $b'$ uniformly on $\mathbb{R}$ as $\varepsilon$ tends to 0. As a consequence of (2.2), we have

$$b'_\varepsilon(r) \geq \frac{\alpha'}{2} \qquad \forall r \in \mathbb{R},$$

for $\varepsilon$ small enough. Then, for a fixed $\varepsilon > 0$ small enough, we denote by $\hat{\theta}^\varepsilon$ (see Lemma 4.1) the unique renormalized solution of (3.5)–(3.7) with $b_\varepsilon$ in place of $b$. The regularity of $\hat{\theta}^\varepsilon$ (see Lemma 4.2) indeed implies that $\hat{\theta}^\varepsilon \in L^1(Q)$. As a consequence we can take $L = L^1(Q)$ in Section 3.
For a fixed $\varepsilon > 0$ small enough, we define the mapping:

$$\psi_1^\varepsilon : L^1(Q) \longrightarrow L^1(Q)$$
$$\theta \longrightarrow \hat{\theta}^\varepsilon = \psi_1^\varepsilon(\theta).$$

The mapping $\psi_1^\varepsilon$ is well defined. In the sequel, we will show that $\psi_1^\varepsilon$ is compact, continuous and that there exists a ball $B$ of $L^1(Q)$ such that $\psi_1^\varepsilon(B) \subset B$.

-i- $\psi_1^\varepsilon$ **is compact**. Let us consider a sequence $\theta_n$, which is bounded in $L^1(Q)$ and define the sequence $\hat{\theta}_n^\varepsilon$ by

$$\psi_1^\varepsilon(\theta_n) = \hat{\theta}_n^\varepsilon.$$

By the definition of $\psi_1^\varepsilon$, for a fixed $n \geq 1$, the functions $u_n$ and $\hat{\theta}_n^\varepsilon$ are the unique solutions of the two problems:

$$\frac{\partial u_n}{\partial t} + (u_n \cdot \nabla)u_n - 2 \text{ div } (\mu(\theta_n)\nabla u_n) = F(\theta_n) \quad \text{in } (H^1_\sigma)'(\Omega), \tag{5.1}$$

$$\text{for almost every } t \in (0,T),$$

$$\text{div } u_n = 0 \quad \text{in } Q, \tag{5.2}$$

$$u_n = 0 \quad \text{on } \Sigma_T, \tag{5.3}$$

$$u_n(t=0) = u_0 \quad \text{in } \Omega. \tag{5.4}$$

and

$$\frac{\partial b_\varepsilon(\hat{\theta}_n^\varepsilon)}{\partial t} + u_n \cdot \nabla b_\varepsilon(\hat{\theta}_n^\varepsilon) - \Delta\hat{\theta}_n^\varepsilon = 2\mu(\theta_n)|Du_n|^2 \quad \text{in } Q, \tag{5.5}$$

$$\hat{\theta}_n^\varepsilon = 0 \quad \text{on } \Sigma_T, \tag{5.6}$$

$$b_\varepsilon(\hat{\theta}_n^\varepsilon)(t=0) = b_\varepsilon(\theta_0) \quad \text{in } \Omega, \tag{5.7}$$

($u_n$ is the usual weak solution of the Navier-Stokes equations (5.1)–(5.4) and $\hat{\theta}_n^\varepsilon$ is the unique renormalized solution of (5.5)–(5.7) given by Lemma 4.1).

Recalling the usual energy equation on the Navier-Stokes equations (5.1)–(5.4) (which is obtained through using $u_n$ as a test function in these equations) gives

$$\frac{1}{2}\int_\Omega |u_n(t)|^2\,dx + \frac{1}{2}\int_0^T\int_\Omega \mu(\theta_n)|Du_n|^2\,dx\,dt \tag{5.8}$$

$$= \int_0^T\int_\Omega F(\theta_n)\cdot u_n\,dx\,dt + \frac{1}{2}\int_\Omega |u_0|^2\,dx.$$

Using assumption (2.3), Poincaré's inequality and Korn's inequality then lead to

$$\int_\Omega |u_n(t)|^2\,dx + \int_0^T\int_\Omega |\nabla u_n|^2\,dx\,dt \leq C\big(\|F(\theta_n)\|^2_{(L^2(Q))^2} + \|u_0\|^2_{(L^2(\Omega))^2}\big)$$

where $C$ is a constant independent of $n$.

Due to the bounded character of $F$ ($\alpha = 0$), indeed the sequence $F(\theta_n)$ is bounded in $(L^\infty(Q))^2$. We obtain the usual estimates (see e.g [11], [25] and [26]):

$$u_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)), \tag{5.9}$$

$$\frac{\partial u_n}{\partial t} \text{ is bounded in } L^2(0,T;(H^1_\sigma(\Omega))'). \tag{5.10}$$

In view of estimates (5.9) and (5.10), we can extract a subsequence (still indexed by $n$) such that:

$$u_n \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{5.11}$$

$$u_n \to u \text{ strongly in } L^2_\sigma(Q), \tag{5.12}$$

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } L^2(0,T;(H^1_\sigma)'(\Omega)),$$

as $n$ tends to $+\infty$, where $u$ is a function of $L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega))$. It implies that:

$$\mu(\theta_n)|Du_n|^2 \text{ is bounded in } L^1(Q). \tag{5.13}$$

In view of (5.13) and Lemma 4.2, we obtain:

$$\hat{\theta}^\varepsilon_n \text{ is bounded in } L^p(Q) \qquad \forall p \in [1,2[. \tag{5.14}$$

Estimate (5.13) and Lemma 4.1 imply that, for a subsequence still indexed by $n$, there exists a measurable function $\vartheta$ such that:

$$\hat{\theta}^\varepsilon_n \longrightarrow \vartheta \text{ almost everywhere in } Q, \tag{5.15}$$

$$b(\hat{\theta}^\varepsilon_n) \longrightarrow b(\vartheta) \text{ almost everywhere in } Q,$$

$$T_K(\hat{\theta}^\varepsilon_n) \rightharpoonup T_K(\vartheta) \text{ in } L^2(0,T;H^1_0(\Omega)),$$

as $n$ tends to $+\infty$ for any $K \geq 0$. In view of (5.14) and (5.15), we conclude that:

$$\hat{\theta}^\varepsilon_n \text{ belongs to a compact set of } L^p(Q),$$

for every $p$ such that $1 \leq p < 2$, so that $\psi^\varepsilon_1 : L^1(Q) \longrightarrow L^1(Q)$ is a compact mapping.

-ii- $\psi^\varepsilon_1$ **is continuous**. Let us consider a sequence $\theta_n$, which belongs to $L^1(Q)$ such that:

$$\theta_n \to \theta, \tag{5.16}$$

strongly in $L^1(Q)$ as $n$ tends to $+\infty$, where $\theta$ is a function of $L^1(Q)$. Let $\hat{\theta}^\varepsilon_n$ and $\hat{\theta}^\varepsilon$ be defined by:

$$\psi^\varepsilon_1(\theta_n) = \hat{\theta}^\varepsilon_n \text{ and } \psi^\varepsilon_1(\theta) = \hat{\theta}^\varepsilon.$$

The sequence $u_n$ is defined as in Step -i- and the function $u$ is the limit defined in (5.11)–(5.12). Since, due to (5.16)

$$F(\theta_n) \to F(\theta) \text{ in } (L^2(Q))^2, \tag{5.17}$$

$$\mu(\theta_n) \to \mu(\theta) \text{ a.e. and in } L^\infty(Q) \text{ weak-*}, \tag{5.18}$$

as $n$ tends to $+\infty$, we can pass to the limit in (5.1)–(5.4) and $u$ is indeed the solution of (3.1)–(3.4). Furthermore, it is well known that, since $N = 2$, $u$ satisfies the following energy equation

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{1}{2}\int_0^T\!\!\int_\Omega \mu(\theta)|Du|^2\,dx\,dt \tag{5.19}$$
$$= \int_0^T\!\!\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\int_\Omega |u_0|^2\,dx.$$

Passing to the limit in (5.8) and comparing with (5.19) we obtain (using (5.18) and the fact that $u_n$ is compact in $(L^2(Q))^2$),

$$u_n \to u \text{ strongly in } L^2(0,T;H^1_\sigma(\Omega)),$$

as $n$ tends to $+\infty$. Then

$$\mu(\theta_n)|Du_n|^2 \to \mu(\theta)|Du|^2 \text{ strongly in } L^1(Q), \tag{5.20}$$

as $n$ tends to $+\infty$. With the help of Lemma 4.1, we conclude that

$$\hat\theta^\varepsilon_n \longrightarrow \hat\theta^\varepsilon \text{ a.e. in } Q, \tag{5.21}$$

$$T_K(\hat\theta^\varepsilon_n) \to T_K(\hat\theta^\varepsilon) \text{ in } L^2(0,T;H^1_0(\Omega)),$$

as $n$ tends to $\infty$ for a fixed $\varepsilon > 0$ and for any $K > 0$, where $\hat\theta^\varepsilon$ is the unique renormalized solution of (3.5)–(3.7) (with $b_\varepsilon$ in place of $b$). In view of (5.14) and (5.21), we have

$$\hat\theta^\varepsilon_n \to \hat\theta^\varepsilon$$

strongly in $L^p(Q)$, for all $p$ such that $1 \le p < 2$. As a consequence

$$\hat\theta^\varepsilon_n \to \hat\theta^\varepsilon$$

strongly in $L^1(Q)$.

-iii- **There exists a ball $B$ of $L^1(Q)$ such that $\psi^\varepsilon_1(B) \subset B$.** We show that there exists a real positive number $R$ such that:

$$\psi^\varepsilon_1(L^1(Q)) \subset B_{L^1(Q)}(0,R).$$

Let $\theta$ be in $L^1(Q)$ and $u \in L^2(0,T;H^1_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))$ be the unique solution of (3.1)–(3.4). We have as in Step -ii-

$$\int_0^T\int_\Omega |Du|^2\,dx\,dt \le C\big(\|F(\theta)\|^2_{(L^2(Q))^2} + \|u_0\|^2_{(L^2(\Omega))^2}\big), \tag{5.22}$$

where $C$ is a constant independent of $\theta$.

Since $F$ and $\mu$ are bounded, there exists a constant $C$ independent of $\theta$ such that

$$||\mu(\theta)|Du|^2||_{L^1(Q)} \le C,$$

and it follows from Lemma 4.2 that there exists a constant $C$ independent of $\theta$ such that

$$\|\hat{\theta}^\varepsilon\|_{L^1(Q)} \le C.$$

For a fixed $\varepsilon > 0$ small enough, Schauder's fixed-point theorem and the definition of $\psi_1^\varepsilon$, permit to conclude that there exists a weak-renormalized solution $(\theta^\varepsilon, u^\varepsilon)$ of the following regularized problem:

$$\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon\cdot\nabla)u^\varepsilon - 2\ \mathrm{div}\ (\mu(\theta^\varepsilon)Du^\varepsilon) = F(\theta^\varepsilon) \quad \text{in } (H^1_\sigma)'(\Omega), \tag{5.23}$$

$$\text{for almost every } t\in(0,T),$$

$$\frac{\partial b_\varepsilon(\theta^\varepsilon)}{\partial t} + u^\varepsilon\cdot\nabla b_\varepsilon(\theta^\varepsilon) - \Delta\theta^\varepsilon = 2\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \quad \text{in } Q, \tag{5.24}$$

$$\mathrm{div}\ u^\varepsilon = 0 \quad \text{in } Q, \tag{5.25}$$

$$u^\varepsilon = 0 \text{ and } \theta^\varepsilon = 0 \quad \text{on } \Sigma_T, \tag{5.26}$$

$$u^\varepsilon(t=0) = u_0 \text{ and } b_\varepsilon(\theta^\varepsilon)(t=0) = b_\varepsilon(\theta_0) \quad \text{in } \Omega. \tag{5.27}$$

Since the function $F$ is bounded on $\mathbb{R}$, it follows that

$$u^\varepsilon \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)),$$

$$\frac{\partial u^\varepsilon}{\partial t} \text{ is bounded in } L^2(0,T;(H^1_\sigma(\Omega))').$$

Upon extracting a subsequence we have

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)),$$

$$u^\varepsilon \to u \text{ strongly in } L^2_\sigma(Q),$$

as $\varepsilon$ tends to 0, where $u$ is a function of $L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$. It implies that:

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \text{ is bounded in } L^1(Q),$$

and then (see Lemma 4.1) again for a subsequence still indexed by $\varepsilon$

$$\theta^\varepsilon \longrightarrow \theta \text{ almost everywhere in } Q,$$

$$b_\varepsilon(\theta^\varepsilon) \longrightarrow b(\theta) \text{ almost everywhere in } Q,$$

$$T_K(\theta^\varepsilon) \rightharpoonup T_K(\theta) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

as $\varepsilon$ tends to 0, where $\theta$ is a measurable function defined on $Q$. It follows that $u$ is the solution of the Navier-Stokes equations (2.15)–(2.16) and that, proceeding as in Step -ii-,

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \to \mu(\theta)|Du|^2 \text{ in } L^1(Q),$$

as $\varepsilon$ tends to 0. In view of Lemma 4.1, this implies that

$$T_K(\theta^\varepsilon) \to T_K(\theta) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

as $\varepsilon$ tends to 0, where $\theta$ is a renormalized solution of (3.5)–(3.7). As a consequence, there exists a weak-renormalized solution $(\theta, u)$ of the problem (2.7)–(2.11)

**CASE 2:** $0 < 2\alpha \leq 1$**.**

Let us proceed by approximation and passage to the limit. We replace the function $F$ by $F^\varepsilon = F \circ T_{\frac{1}{\varepsilon}}$, for $\varepsilon > 0$, and we consider the following approximate problem

$$\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla)u^\varepsilon - 2 \text{ div } (\mu(\theta^\varepsilon)Du^\varepsilon) = F^\varepsilon(\theta^\varepsilon) \quad \text{in } (H_\sigma^1(\Omega))', \tag{5.28}$$
$$\text{for almost every } t \in (0,T),$$

$$\frac{\partial b(\theta^\varepsilon)}{\partial t} + u^\varepsilon \cdot \nabla b(\theta^\varepsilon) - \Delta\theta^\varepsilon = 2\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \quad \text{in } Q, \tag{5.29}$$

$$\text{div } u^\varepsilon = 0 \quad \text{in } Q, \tag{5.30}$$

$$u^\varepsilon = 0 \text{ and } \theta^\varepsilon = 0 \quad \text{on } \Sigma_T, \tag{5.31}$$

$$u^\varepsilon(t=0) = u_0 \text{ and } b(\theta^\varepsilon)(t=0) = b(\theta_0) \quad \text{in } \Omega. \tag{5.32}$$

The function $F^\varepsilon$ being continuous and bounded, we apply the result of Case 1, so that there exists a weak-renormalised solution $(\theta^\varepsilon, u^\varepsilon)$ of the approximate system (5.28)–(5.32). Using estimate (4.10) for $\theta^\varepsilon$, we have

$$\int_\Omega |\theta^\varepsilon|(t)dx \leq \frac{1}{\alpha} (\|\mu(\theta^\varepsilon)|Du^\varepsilon|^2\|_{L^1(Q)} + \|b(\theta_0)\|_{L^1(\Omega)}).$$

Now, estimate (5.22) for $u^\varepsilon$ gives

$$\int_0^T\int_\Omega |Du^\varepsilon|^2 \, dx \, dt \leq C\big(\|F(\theta^\varepsilon)\|^2_{(L^2(Q))^2} + \|u_0\|^2_{(L^2(\Omega))^2}\big),$$

where $C$ is a constant independent of $\varepsilon$. Since $\mu$ is bounded, using the growth condition (2.4) on $F$ implies that

$$\int_\Omega |\theta^\varepsilon|(t)dx \le c_1 \int_0^t\!\!\int_\Omega |\theta^\varepsilon|^{2\alpha} dx\, dt + c_2,$$

where $c_1$ and $c_2$ are two constants which do not depend on $\varepsilon$.
Since $0 < 2\alpha \le 1$, Gronwall's lemma shows that $(\theta^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and as a consequence $(F^\varepsilon(\theta^\varepsilon))_{\varepsilon>0}$ is bounded in $(L^2(Q))^2$. Then

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \text{ is bounded in } L^1(Q).$$

Proceeding as in Step -i-, we deduce that for a subsequence still indexed by $\varepsilon$

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)),$$

$$\theta^\varepsilon \longrightarrow \theta \text{ almost everywhere in } Q.$$

It follows that $F^\varepsilon(\theta^\varepsilon)$ converges weakly to $F(\theta)$ in $(L^2(Q))^2$ and then $u$ is the solution of (2.15)–(2.16) and we have

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \to \mu(\theta)|Du|^2 \text{ strongly in } L^1(Q).$$

Applying Lemma 4.1, we deduce that

$$T_K(\theta^\varepsilon) \to T_K(\theta) \text{ in } L^2(0,T;H^1_0(\Omega)),$$

as $\varepsilon$ tends to 0, where $\theta$ is a renormalized solution of (3.5)–(3.7). Thus, $(\theta,u)$ is a weak-renormalized solution of the problem (2.7)–(2.11).

**CASE 3:** $1 < 2\alpha < 2$.

For a fixed $\theta \in L^{2\alpha}(Q)$, due to the growth assumption (2.4) on $F$, $F(\theta) \in (L^2(Q))^2$ and again there exists a unique weak solution $u$ of (3.1)–(3.4) in $L^2(0,T;H^1_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))$. As in the case $\alpha = 0$, for $\varepsilon > 0$ small enough, there exists a unique renormalized solution $\hat\theta^\varepsilon$ of (3.5)–(3.7) with $b_\varepsilon$ in place of $b$. The regularity of $\hat\theta^\varepsilon$ (see Lemma 4.2) implies that $\hat\theta^\varepsilon \in L^{2\alpha}(Q)$ because $1 < 2\alpha < 2$. As a consequence, we can take $L = L^{2\alpha}(Q)$ in the fixed-point argument of Section 3.
For a fixed $\varepsilon > 0$ small enough, we define the mapping:

$$\psi_2^\varepsilon : L^{2\alpha}(Q) \longrightarrow L^{2\alpha}(Q)$$

$$\theta \longrightarrow \hat\theta^\varepsilon = \psi_2^\varepsilon(\theta)$$

We show that $\psi_2^\varepsilon$ is compact, continuous and that there exists a ball $B$ of $L^{2\alpha}(Q)$ such that $\psi_2^\varepsilon(B) \subset B$.

-i- $\psi_2^\varepsilon$ **is compact**. Let us consider a bounded sequence $\theta_n$ in $L^{2\alpha}(Q)$ and define the sequence $\hat\theta_n^\varepsilon$ by

$$\psi_2^\varepsilon(\theta_n) = \hat\theta_n^\varepsilon.$$

For a fixed $n \geq 1$, by definition of $\psi_2^\varepsilon$ the functions $u_n$ and $\hat\theta_n^\varepsilon$ are respectively the unique solutions of the two problems (5.1)–(5.4) and (5.5)–(5.7). Due to the growth condition (2.4) on $F$, the sequence $F(\theta_n)$ is bounded in $(L^2(Q))^2$ and then

$$u_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)), \tag{5.33}$$

$$\frac{\partial u_n}{\partial t} \text{ is bounded in } L^2(0,T;(H^1_\sigma)'(\Omega)).$$

This implies that

$$\mu(\theta_n)|Du_n|^2 \text{ is bounded in } L^1(Q). \tag{5.34}$$

Proceeding as above with the help of Lemma 4.1 and Lemma 4.2, we deduce that

$$\hat\theta_n^\varepsilon \text{ is bounded in } L^p(Q) \qquad \forall p \in [1,2[. \tag{5.35}$$

$$\hat\theta_n^\varepsilon \longrightarrow \vartheta \text{ almost everywhere in } Q, \tag{5.36}$$

as $n$ tends to $+\infty$. Since $1 < 2\alpha < 2$ from (5.35) and (5.36) we conclude that

$$\hat\theta_n^\varepsilon \text{ belongs to a compact set of } L^{2\alpha}(Q) \tag{5.37}$$

and $\psi_2^\varepsilon$ is compact.

-ii- $\psi_2^\varepsilon$ **is continuous**. Let us consider a sequence $\theta_n$ of $L^{2\alpha}(Q)$ such that

$$\theta_n \to \theta,$$

strongly in $L^{2\alpha}(Q)$ as $n$ tends to $+\infty$, where $\theta$ is a function of $L^{2\alpha}(Q)$. Let $\hat\theta_n^\varepsilon$ and $\hat\theta^\varepsilon$ be defined by:

$$\psi_2^\varepsilon(\theta_n) = \hat\theta_n^\varepsilon \text{ and } \psi_2^\varepsilon(\theta) = \hat\theta^\varepsilon.$$

Since

$$F(\theta_n) \rightharpoonup F(\theta) \text{ in } (L^2(Q))^2,$$

as $n$ tends to $+\infty$, the corresponding sequence $u_n$ given by (5.1)–(5.4) is compact in $(L^2(Q))^2$. We can repeat exactly the same argument that led to (5.20) to show that:

$$\mu(\theta_n)|Du_n|^2 \to \mu(\theta)|Du|^2 \text{ strongly in } L^1(Q),$$

as $n$ tends to $+\infty$. By Lemma 4.1, we deduce that

$$\hat\theta_n^\varepsilon \longrightarrow \hat\theta^\varepsilon \text{ a.e. in } Q, \tag{5.38}$$

$$T_K(\hat{\theta}_n^\varepsilon) \to T_K(\hat{\theta}^\varepsilon) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

as $n$ tends to $\infty$ for a fixed $\varepsilon > 0$ and for any $K > 0$ , where $\hat{\theta}^\varepsilon$ is the unique renormalized solution of (3.5)–(3.7) (with $b_\varepsilon$ in place of $b$). Since $1 < 2\alpha < 2$, (5.35) and (5.38) give

$$\hat{\theta}_n^\varepsilon \to \hat{\theta}^\varepsilon$$

strongly in $L^{2\alpha}(Q)$.

-iii- **There exists a ball** $B$ **of** $L^{2\alpha}(Q)$ **such that** $\psi_2^\varepsilon(B) \subset B$. Let $R$ be a positive real number. We will show that if the data are small enough, there exists $R_0 > 0$ such that:

$$\psi_2^\varepsilon(B_{L^{2\alpha}(Q)}(0,R_0)) \subset B_{L^{2\alpha}(Q)}(0,R_0).$$

We assume that $\theta$ belongs to $B_{L^{2\alpha}(Q)}(0,R)$. In what follows, $C$ denotes a generic constant which depends on $\Omega$, $T$, $m_1$ and $m_0$. We recall that $u$, which belongs to $L^2(0,T;H_\sigma^1(\Omega)) \cap L^\infty(0,T;L_\sigma^2(\Omega))$ is the unique solution of the problem (3.1)–(3.4), then we use $u$ as a test function in (3.1), we obtain:

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{1}{2}\int_0^T\!\!\int_\Omega \mu(\theta)|Du|^2\,dx\,dt$$
$$= \int_0^T\!\!\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\int_\Omega |u_0|^2\,dx,$$

then

$$\frac{1}{2}\int_\Omega |u(t)|^2\,dx + \frac{m_0}{2}\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt$$
$$\le \int_0^T\!\!\int_\Omega F(\theta)\cdot u\,dx\,dt + \frac{1}{2}\|u_0\|^2_{(L^2(\Omega))^2},$$

which implies that

$$m_0\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt \le 2\int_0^T \|F(\theta)\|_{(L^2(\Omega))^2}\|u\|_{(L^2(\Omega))^2}\,dt + \|u_0\|^2_{L^2(\Omega)}. \tag{5.39}$$

In what follows, $C$ denotes a constant independent upon $\varepsilon$, $\theta$, $F$ and $u_0$. Inequality (5.39) and Poincaré's inequality lead to:

$$m_0\int_0^T\!\!\int_\Omega |Du|^2\,dx\,dt \le C\int_0^T \|F(\theta)\|_{(L^2(\Omega))^2}\|\nabla u\|_{(L^2(\Omega))^2}\,dt + \|u_0\|^2_{(L^2(\Omega))^2}, \tag{5.40}$$

Young's inequality, (5.40) and Korn's inequality permit us to deduce that:

$$\int_0^T\int_\Omega |Du|^2\,dx\,dt \le C\big(\|F(\theta)\|^2_{(L^2(Q))^2} + \|u_0\|^2_{(L^2(\Omega))^2}\big). \tag{5.41}$$

In view of Lemma 4.2 and (5.41), we obtain

$$\|\hat{\theta}^\varepsilon\|_{L^p(Q)} \le C\left[\|F(\theta)\|^2_{(L^2(Q))^2} + \|u_0\|^2_{(L^2(\Omega))^2} + \|b_\varepsilon(\theta_0)\|_{L^1(\Omega)}\right], \tag{5.42}$$

for all $p$ such that $1 \le p < 2$. By the growth assumption on $F$, we have:

$$|F(\theta)|^2 \le 2(a^2 + M^2|\theta|^{2\alpha}) \text{ a.e. in } Q,$$

and then

$$\|F(\theta)\|^2_{(L^2(Q))^2} \le 2a^2 \text{ meas}(\Omega)T + 2M^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)}. \tag{5.43}$$

It follows that from (5.42) and (5.43):

$$\|\hat{\theta}^\varepsilon\|_{L^p(Q)} \le C\left[a^2 \text{ meas}(\Omega)T + M^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)}\right.$$

$$\left. + C\|u_0\|^2_{(L^2(\Omega))^2} + \|b_\varepsilon(\theta_0)\|_{L^1(\Omega)}\right],$$

for all $p$ such that $1 \le p < 2$. Because $1 < 2\alpha < 2$, we deduce that:

$$\|\hat{\theta}^\varepsilon\|_{L^{2\alpha}(Q)} \le C\left[a^2 + M^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)} + \|u_0\|^2_{(L^2(\Omega))^2} + \|b_\varepsilon(\theta_0)\|_{L^1(\Omega)}\right].$$

Since the sequence $b_\varepsilon(\theta_0)$ converges to $b(\theta_0)$ in $L^1(\Omega)$ as $\varepsilon$ tends to 0, it follows that for example

$$\|\hat{\theta}^\varepsilon\|_{L^{2\alpha}(Q)} \le C\left[a^2 + M^2\|\theta\|^{2\alpha}_{L^{2\alpha}(Q)} + \|u_0\|^2_{(L^2(\Omega))^2} + 2\|b(\theta_0)\|_{L^1(\Omega)}\right], \tag{5.44}$$

for $\varepsilon$ small enough.

Now there exists a positive real number $\eta > 0$ and a positive real number $R(\eta) > 0$, which do not depend upon $\varepsilon$, such that if

$$a^2 + \|u_0\|^2_{(L^2(\Omega))^2} + \|b(\theta_0)\|_{L^1(\Omega)} \le \eta, \tag{5.45}$$

then

$$C\left[a^2 + M^2R(\eta)^{2\alpha} + \|u_0\|^2_{(L^2(\Omega))^2} + 2\|b(\theta_0)\|_{L^1(\Omega)}\right] \le R(\eta).$$

As a consequence of (5.44) we conclude that if (5.45) holds true then

$$\psi_2^\varepsilon(B_{L^{2\alpha}(Q)}(0, R(\eta))) \subset B_{L^{2\alpha}(Q)}(0, R(\eta)).$$

Schauder's fixed-point theorem and the definition of $\psi_2^\varepsilon$ imply that under the condition (5.45) and for $\varepsilon$ small enough, there exists a weak-renormalized solution $(\theta^\varepsilon, u^\varepsilon)$ of the following problem:

$$\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla)u^\varepsilon - 2 \text{ div } (\mu(\theta^\varepsilon)Du^\varepsilon) = F(\theta^\varepsilon) \quad \text{in } (H^1_\sigma)'(\Omega), \tag{5.46}$$
$$\text{for almost every } t \in (0,T),$$
$$\frac{\partial b_\varepsilon(\theta^\varepsilon)}{\partial t} + u^\varepsilon \cdot \nabla b_\varepsilon(\theta^\varepsilon) - \Delta\theta^\varepsilon = 2\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \quad \text{in } Q, \tag{5.47}$$
$$\text{div } u^\varepsilon = 0 \quad \text{in } Q, \tag{5.48}$$
$$u^\varepsilon = 0 \text{ and } \theta^\varepsilon = 0 \quad \text{on } \Sigma_T, \tag{5.49}$$
$$u^\varepsilon(t=0) = u_0 \text{ and } b_\varepsilon(\theta^\varepsilon)(t=0) = b_\varepsilon(\theta_0) \quad \text{in } \Omega, \tag{5.50}$$

such that:

$$\|\theta^\varepsilon\|_{L^{2\alpha}(Q)} \leq R(\eta).$$

We now pass to the limit with respect to $\varepsilon$ in (5.46)–(5.50). Due to (2.4) and from the above estimate the sequence $F(\theta^\varepsilon)$ is bounded in $(L^2(Q))^2$ and we end the proof as in the case $0 < 2\alpha \leq 1$. As a consequence, under the condition (5.45), there exists a weak-renormalized solution $(\theta, u)$ of the problem (2.7)–(2.11).

**CASE 4:** $2 \leq 2\alpha < 3$.

Under this assumption on $\alpha$, the fonction $F(\theta)$ can not be expected in $(L^2(Q))^2$ and we will use the uncoupled regularity of $\theta$ with respect to $t$ and $x$ given by Lemma 4.2. Let $q > \alpha$ be a real number. For $\theta \in L^{2\alpha}(0,T; L^q(\Omega))$ the growth assumption (2.4) implies that $F(\theta)$ belongs to $L^2(0,T;(L^p(\Omega))^2)$ for any real number $1 < p < \frac{q}{\alpha}$. Since $N = 2$, Sobolev's embedding then gives that

$$F(\theta) \in L^2(0,T;(H^{-1}(\Omega))^2) \tag{5.51}$$

with

$$\|F(\theta)\|_{L^2(0,T;(H^{-1}(\Omega))^2)} \leq C(a + \|\theta\|^\alpha_{L^r(0,T;L^q(\Omega))}). \tag{5.52}$$

As a consequence of (5.51), for any $\theta \in L^{2\alpha}(0,T;L^q(\Omega))$, the problem (3.1)–(3.4) admits a unique solution $u$ in $L^2(0,T;H^1_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))$. Then, by Lemma 4.1, the parabolic problem (3.5)–(3.7) with $b$ replaced by $b_\varepsilon$ admits a unique renormalized solution $\hat{\theta}^\varepsilon$ which satisfies the regularity of Lemma 4.2. As a consequence, in order to insure that $\hat{\theta}^\varepsilon$ belongs to the same space $L^{2\alpha}(0,T;L^q(\Omega))$ than $\theta$, it is sufficient to choose $\alpha < q < \frac{2\alpha}{2\alpha-1}$ which is possible since $2\alpha < 3$. This leads to the choice $L = L^r(0,T;L^q(\Omega))$ in the process described in Section 3 and to consider the mapping $\psi_3^\varepsilon$ for a fixed $\varepsilon$ (small enough) defined by

$$\psi_3^\varepsilon : L^r(0,T;L^q(\Omega)) \longrightarrow L^r(0,T;L^q(\Omega))$$
$$\theta \longrightarrow \hat{\theta}^\varepsilon = \psi_3^\varepsilon(\theta).$$

In the sequel, we will show that $\psi_3^\varepsilon$ is compact, continuous and that there exists a ball $B$ of $L^r(0,T;L^q(\Omega))$ such that $\psi_3^\varepsilon(B) \subset B$.

-i-$\psi_3^\varepsilon$ **is compact**. Let us consider a sequence $\theta_n$ which is bounded in $L^r(0,T;L^q(\Omega))$ and define the sequence $\hat{\theta}_n^\varepsilon$ by

$$\psi_3^\varepsilon(\theta_n) = \hat{\theta}_n^\varepsilon.$$

For a fixed $n \geq 1$, the functions $u_n$ and $\hat{\theta}_n^\varepsilon$ are respectively the unique solutions of the two problems (5.1)–(5.4) and (5.5)–(5.7).

Since the sequence $\theta_n$ is bounded in $L^r(0,T;L^q(\Omega))$, for any real number $1 < p < \frac{q}{\alpha}$ the sequence $F(\theta_n)$ is bounded in $L^2(0,T;(L^p(\Omega))^2)$ and then in $L^2(0,T;(H^{-1}(\Omega))^2)$ so that the sequence $u_n$ satisfies the estimates (5.33)–(5.34) of the step -i- of Case 3. Using Lemmas 4.1 and 4.2 permits to obtain

$$\hat{\theta}_n^\varepsilon \longrightarrow \vartheta \text{ almost everywhere in } Q, \tag{5.53}$$

$$\hat{\theta}_n^\varepsilon \text{ is bounded in } L^{r_1}(0,T;L^{q_1}(\Omega)), \tag{5.54}$$

for any couple $(q_1, r_1)$ such that $1 < q_1 < \infty$ and $1 \leq r_1 < \frac{q_1}{q_1-1}$ and where $\vartheta$ is a measurable function defined on $Q$. Since we are at liberty to choose $q < q_1$ and $2\alpha < r_1 < \frac{q_1}{q_1-1}$, we deduce that from (5.53) and (5.54)

$$\hat{\theta}_n^\varepsilon \to \vartheta \text{ strongly in } L^r(0,T;L^q(\Omega)),$$

as $n$ tends to $+\infty$ and $\psi_3^\varepsilon$ is a compact mapping.

-ii-$\psi_3^\varepsilon$ **is continuous**. Let us consider a sequence $\theta_n$ of $L^r(0,T;L^q(\Omega))$ such that:

$$\theta_n \to \theta \text{ strongly in } L^r(0,T;L^q(\Omega)),$$

as $n$ tends to $+\infty$, where $\theta$ is a function of $L^r(0,T;L^q(\Omega))$. Let $\hat{\theta}_n^\varepsilon$ and $\hat{\theta}^\varepsilon$ be defined by:

$$\psi_3^\varepsilon(\theta_n) = \hat{\theta}_n^\varepsilon \text{ and } \psi_3^\varepsilon(\theta) = \hat{\theta}^\varepsilon.$$

Due to the choice of $q$, the assumption (2.4) implies that the sequence $F(\theta_n)$ is compact in $L^2(0,T;(L^p(\Omega))^2)$ for any real number $1 < p < \frac{q}{\alpha}$. Since the embedding $L^2(0,T;L^p(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$ is continuous it follows that

$$F(\theta_n) \to F(\theta) \text{ in } L^2(0,T;(H^{-1}(\Omega))^2). \tag{5.55}$$

as $n$ tends to $+\infty$. Now we can repeat the argument of the step -ii- of the case $\alpha = 0$ by using (5.55) instead of (5.17) to pass to the limit in (5.8) and we still have

$$\mu(\theta_n)|Du_n|^2 \to \mu(\theta)|Du|^2 \text{ in } L^1(Q),$$

as $n$ tends to $+\infty$. We conclude that by Lemma 4.1 and Lemma 4.2

$$\hat{\theta}_n^\varepsilon \to \hat{\theta}^\varepsilon \text{ strongly in } L^r(0,T;L^q(\Omega)),$$

as $n$ tends to $\infty$, where for a fixed $\varepsilon > 0$, $\hat{\theta}^\varepsilon$ is the unique renormalized solution of (3.5)–(3.7) (with $b_\varepsilon$ in place of $b$). Then $\psi_3^\varepsilon$ is a continuous mapping.

-iii- **There exists a ball** $B$ **of** $L^r(0,T;L^q(\Omega))$ **such that** $\psi_3^\varepsilon(B) \subset B$. We show that there exists a positive real number $\eta > 0$ and a positive real number $R(\eta) > 0$, which do not depend upon $\varepsilon$, such that if

$$a^2 + \|u_0\|^2_{(L^2(\Omega))^2} + \|b(\theta_0)\|_{L^1(\Omega)} \le \eta, \tag{5.56}$$

then

$$\psi_3^\varepsilon(B_{L^{2\alpha}(Q)}(0,R(\eta))) \subset B_{L^{2\alpha}(Q)}(0,R(\eta)).$$

We proceed as in the step -iii- of the case $1 \le 2\alpha \le 2$ upon replacing $\|F(\theta)\|_{(L^2(Q))^2}$ by $\|F(\theta)\|_{L^2(0,T;(H^{-1}(\Omega))^2)}$ and we obtain

$$\int_0^T\int_\Omega |Du|^2\,dx\,dt \le C\big(\|F(\theta)\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} + \|u_0\|^2_{(L^2(\Omega))^2}\big).$$

Appealing now to Lemma 4.2 and to (5.52) gives

$$\|\hat{\theta}^\varepsilon\|_{L^r(0,T;L^q(\Omega))} \le C\left[a^2 + M^2\|\theta\|^{2\alpha}_{L^r(0,T;L^q(\Omega))} + \|u_0\|^2_{(L^2(\Omega))^2} + \|b_\varepsilon(\theta_0)\|_{L^1(\Omega)}\right],$$

with $C$ is a constant independent of $\varepsilon$, $\|\theta\|_{L^r(0,T;L^q(\Omega))}$, $u_0$, $M$ and $\theta_0$. Then the proof of the result is identical to that of the step -iii- of the case $1 \le 2\alpha \le 2$.

Schauder's fixed-point theorem and the definition of $\psi_3^\varepsilon$, permit us to conclude that under the condition (5.56), there exists a weak-renormalized solution $(\theta^\varepsilon, u^\varepsilon)$ of the regularized problem (5.46)–(5.50) which satisfies

$$\|\theta^\varepsilon\|_{L^r(0,T;L^q(\Omega))} \le R(\eta),$$

for $\varepsilon$ small enough. With (5.52), we obtain

$$F(\theta^\varepsilon) \text{ is bounded in } L^2(0,T;(H^{-1}(\Omega))^2),$$

so that

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \text{ is bounded in } L^1(Q).$$

As a consequence of Lemma 4.1 and Lemma 4.2 we deduce that for a subsequence still indexed by $\varepsilon$

$$\theta^\varepsilon \longrightarrow \theta \text{ almost everywhere in } Q,$$
$$b_\varepsilon(\theta^\varepsilon) \longrightarrow b(\theta) \text{ almost everywhere in } Q,$$
$$T_K(\theta^\varepsilon) \rightharpoonup T_K(\theta) \text{ in } L^2(0,T;H_0^1(\Omega)),$$
$$\|\theta^\varepsilon\|_{L^{r_1}(0,T;L^{q_1}(\Omega))} \leq C(\eta),$$

where $\theta$ belongs to $L^{r_1}(0,T;L^{q_1}(\Omega))$ for any $(q_1,r_1)$ such that $q_1 > 1$ and $1 \leq r_1 < \frac{q_1}{q_1-1}$.
Proceeding as in the proof of the compactness of $\psi_3^\varepsilon$ leads to

$$\theta^\varepsilon \to \theta \text{ in } L^{2\alpha}(0,T;L^q(\Omega)),$$
$$F(\theta^\varepsilon) \to F(\theta) \text{ in } L^2(0,T;(H^{-1}(\Omega))^2),$$

and

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \to \mu(\theta)|Du|^2 \text{ in } L^1(Q),$$

as $\varepsilon$ tends to 0. Using again Lemma 4.1 permits us to conclude that: $\theta$ is a renormalized solution of (3.5)–(3.7). As a consequence, there exists a weak-renormalized solution $(\theta,u)$ of the problem (2.7)–(2.11).

□

## 6. Existence of a solution for N=3

In this section, we assume that $F$ is a continuous and bounded function from $\mathbb{R}$ into $\mathbb{R}^3$.

**Theorem 6.1.** *Assume that (2.1), (2.2), (2.3) and (2.6) hold true. Assume that $F$ is a continuous and bounded function from $\mathbb{R}$ into $\mathbb{R}^3$, and $u_0 \in (H_0^1(\Omega))^3$ such that $\operatorname{div} u_0 = 0$ and $u_0 \cdot n = 0$ on $\partial\Omega$. There exists a real positive number $\eta$ such that if $\|u_0\|_{(H_0^1(\Omega))^3} + \|F\|_{(L^\infty(\mathbb{R}))^3} \leq \eta$, then there exists at least a weak-renormalized solution of the system (2.7)–(2.11) for $N=3$ (in the sense of Definition 2.1).*

*Proof of Theorem 6.1.* Since the proof relies on similar techniques to the ones developed in the previous sections, we just point out how to modify the arguments.
In a first step, we assume that $b'$ is locally Lipschitz continuous. The fixed-point space is $L = L^1(Q)$. For a fixed $\theta$ in $L^1(Q)$, it is known that there exists $\eta > 0$ such that if $\|u_0\|_{(H_0^1(\Omega))^3} + \|F\|_{(L^\infty(\mathbb{R}))^3} \leq \eta$, then the Navier-Stokes

equations (2.7)–(2.11) admit a unique solution $u \in L^\infty(0,T;(H_0^1(\Omega))^3) \cap L^2(0,T;(H^2(\Omega))^3)$ (see Theorem 3.11 of [26]). The unique renormalized solution $\hat{\theta}$ of (3.5)–(3.7) indeed belongs to $L^1(Q)$ (see Lemma 4.2).
We denote by $\psi_4$ the mapping defined by:

$$\psi_4 : L^1(Q) \longrightarrow L^1(Q)$$

$$\theta \longrightarrow \hat{\theta} = \psi_4(\theta)$$

By the same arguments used in the preceding sections, particularly in the case where $\alpha = 0$, we know that $\psi_4$ satisfies the conditions of the Schauder's fixed-point theorem, which implies the existence of a weak-renormalized solution $(\theta, u)$ of the system (2.7)–(2.11) when $b'$ is locally Lipschitz.
In a second step, we regularize $b$ by $b_\varepsilon$ as in the previous sections. We recall that for a fixed $\varepsilon > 0$ small enough, $b_\varepsilon$ satisfies the assumptions (2.1), (2.2) and $b'_\varepsilon$ is locally Lipschitz.
We consider the following approximate problem:

$$\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla)u^\varepsilon - 2 \text{ div } (\mu(\theta^\varepsilon)Du^\varepsilon) = F(\theta^\varepsilon) \quad \text{in } (H_\sigma^1)'(\Omega), \tag{6.1}$$
$$\text{for almost every } t \in (0,T),$$
$$\frac{\partial b_\varepsilon(\theta^\varepsilon)}{\partial t} + u^\varepsilon \cdot \nabla b_\varepsilon(\theta^\varepsilon) - \Delta\theta^\varepsilon = 2\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \quad \text{in } Q, \tag{6.2}$$
$$\text{div } u^\varepsilon = 0 \quad \text{in } Q, \tag{6.3}$$
$$u^\varepsilon = 0 \text{ and } \theta^\varepsilon = 0 \quad \text{on } \Sigma_T, \tag{6.4}$$
$$u^\varepsilon(t=0) = u_0 \text{ and } b(\theta^\varepsilon)(t=0) = b_\varepsilon(\theta_0) \quad \text{in } \Omega. \tag{6.5}$$

Since $\|u_0\|_{(H_0^1(\Omega))^3} + \|F\|_{(L^\infty(\mathbb{R}))^3} \leq \eta$ and according to the result of first step, we know that for a fixed $\varepsilon > 0$ (small enough), there exists a weak-renormalized solution $(\theta^\varepsilon, u^\varepsilon)$ of problem (6.1)–(6.5). Moreover the following estimates hold true uniformly with respect to $\varepsilon$ (see again Theorem 3.11 of [26]):

$$u^\varepsilon \text{ is bounded in } L^2(0,T;H^2(\Omega)),$$

$$\frac{\partial u^\varepsilon}{\partial t} \text{ is bounded in } L^2(0,T;(H_\sigma^1)'(\Omega)).$$

Thanks to an Aubin's type lemma (see e.g [24]), we may, then, extract a subsequence such that:

$$u^\varepsilon \to u \text{ strongly in } L^2(0,T;H_\sigma^1(\Omega)), \tag{6.6}$$

as $\varepsilon$ tends to 0, where $u$ is a function which belongs to $L^\infty(0,T;(H_0^1(\Omega))^3)\cap L^2(0,T;(H^2(\Omega))^3)$. It implies that:

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \text{ is bounded in } L^1(Q).$$

Then using Lemma 4.1, there exists a subsequence still indexed by $\varepsilon$ such that:

$$\theta^\varepsilon \longrightarrow \theta \text{ almost everywhere in } Q, \tag{6.7}$$

$$T_K(\theta^\varepsilon) \rightharpoonup T_K(\theta) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

as $\varepsilon$ tends to 0, where $\theta$ is a measurable function. In view of (6.6) and (6.7), we deduce that

$$\mu(\theta^\varepsilon)|Du^\varepsilon|^2 \to \mu(\theta)|Du|^2 \text{ in } L^1(Q),$$

as $\varepsilon$ tends to 0. Thanks again to Lemma 4.1, this last result allows to conclude that $\theta$ is a renormalized solution of (3.5)–(3.7). As a consequence, there exists a weak-renormalized solution $(\theta,u)$ of the problem (2.7)–(2.11). □

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