EXISTENCE RESULT FOR NONLINEAR PARABOLIC EQUATIONS WITH LOWER ORDER TERMS

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ABSTRACT. In this paper we prove the existence of a renormalized solution for a class of nonlinear parabolic problems whose prototype is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + \operatorname{div}\left(c\,|u|^{\gamma-1}\,u\right) + b\,|\nabla u|^{\delta} = f - \operatorname{div} g & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $Q_T = \Omega \times (0,T), \Omega$ is an open and bounded subset of $\mathbb{R}^N$, $N \geq 2$, $T > 0$, Δ_p is the so called p-Laplace operator, $\gamma = \frac{(N+2)(p-1)}{N+p}$, $c \in \left(L^r(Q_T)\right)^N$ with $r = \frac{N+p}{p-1}$, $\delta = \frac{N(p-1)+p}{N+2}$, $b \in L^{N+2,1}(Q_T)$, $f \in L^1(Q_T)$, $g \in (L^{p'}(Q_T))^N$ and $u_0 \in L^1(\Omega)$.

1. INTRODUCTION

In the present paper we study a nonlinear parabolic problem whose prototype is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + \operatorname{div}(c|u|^{\gamma-1} u) + b\,|\nabla u|^{\delta} = f - \operatorname{div} g & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Δ_p is the so called p-Laplace operator, Q_T is the cylinder $\Omega \times (0,T)$, Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, $T > 0$, $\gamma = \frac{(N+2)(p-1)}{N+p}$, $c \in (L^r(Q_T))^N$ with $r = \frac{N+p}{p-1}$, $\delta = \frac{N(p-1)+p}{N+2}$, $b \in L^{N+2,1}(Q_T)$, $f \in L^1(Q_T)$, $g \in (L^{p'}(Q_T))^N$ and $u_0 \in L^1(\Omega)$.

We are interested in proving an existence result to (1.1). The difficulties connected to this problem are due to the L^1 data and to the presence of the two terms $\operatorname{div}(c(x,t)|u|^{\gamma-1}u)$ and $b(x,t)|\nabla u|^{\delta}$ which induce a lack of coercivity.

When $b \equiv c \equiv 0$ the existence of weak solutions was proved in [8] (see also [7]) for L^1 data or bounded measure data for $p > 2 - \frac{1}{N+1}$. Problem (1.1) with $c \equiv 0$ has been analyzed in [26] and the existence of a weak solution with $f \in L^1$ is obtained under the same condition on p. In these papers a weak solution belongs to $L^m((0,T); W_0^{1,m}(\Omega))$ with $m < \frac{p(N+1)-N}{N+1}$ and Equation (1.1) is verified in the sense of distributions. It follows that it is natural to impose $1 < \frac{p(N+1)-N}{N+1}$

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which is equivalent to the condition $p > 2 - \frac{1}{N+1}$. Moreover it is well known that this weak solution is not unique in general (see [29] and [27]). To remove this condition on p and to guarantee stability properties we use in the present paper the framework of renormalized solutions.

The notion of renormalized solution was introduced in [16, 17] for first order equations and has been developed for elliptic problems with L^1 data in [22] (see also [23]) and with bounded measure data in [13]. This notion was adapted to parabolic equations with L^1 data in [3, 4] (see also [25] for a definition of renormalized solution to parabolic equation with general measure data). The notion of entropy solution (which is equivalent to the notion of renormalized solution in the L^1 case) developed in [1] may also be used for parabolic equations of type (1.1) (see [28]).

The existence of a renormalized solution for a nonlinear parabolic problem with a lower order term of the type $\operatorname{div}(\Phi(u))$, with Φ continuous function in $\mathbb{R}^N$ has been proved in [6]. When $p = 2$, $f \equiv 0$, $g \equiv 0$, $b \equiv 0$ and $c \in \left(L^2(Q_T)\right)^N$ problem (1.1) is studied in [10] in the framework of entropy solutions. Existence of renormalized solutions for problem (1.1) is proved in [15] when $b \equiv 0$, $g \equiv 0$ with $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$.

In the present paper we prove an existence result for renormalized solutions to a class of problems whose prototype is (1.1) with the two lower order terms. We underline that we don't make any assumptions on the smallness of the coefficients. It is worth noting that for the analogous elliptic equation with two lower order terms (see e.g. [14, 18, 19]) assuming that one of the terms b or c is small enough is necessary to obtain an existence result. The proof consists of several steps. First of all we introduce an approximated problem, then we derive an a priori estimate for the gradients of its solutions following an idea contained in [26] (see also [2], [11]). We consider a partition of the entire interval $[0, T]$ into a finite number of intervals $[0, t_1]$, $[t_1, t_2]$, ..., $\left[t_{n-1}, T\right]$ and in each cylinder $Q_{t_i} = \Omega \times [t_{i-1}, t_i]$ we obtain an a priori bound for the solution and its gradient which allows us to deduce the a priori estimates on the entire cylinder. Such a priori bounds are obtained using a technical lemma ([1] see also [12]) contained in Appendix A. Finally we pass to the limit in the approximated problem.

2. Assumptions and Definitions

In this section we recall the definition of a renormalized solution to nonlinear parabolic problems with lower order terms and $L^1(\Omega\times(0,T))+L^{p'}((0,T);W^{-1,p'}(\Omega))$ data.

More precisely we consider the following problem

$$\begin{cases} u_t - \operatorname{div}(a(x,t,u,\nabla u)) \\ \qquad + \operatorname{div}(K(x,t,u)) + H(x,t,\nabla u) = f - \operatorname{div} g & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where Q_T is the cylinder $\Omega \times (0,T)$, Ω is a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$, $T > 0$, $p > 1$.

We assume that the following assumptions hold true:

- $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that

$$a(x,t,s,\xi)\xi \geq \alpha\,|\xi|^p, \quad \alpha > 0, \tag{2.2}$$

for any $k > 0$ there exists $\beta_k > 0$ and $h_k \in L^{p'}(Q_T)$ such that

$$|a(x,t,s,\xi)| \leq h_k + \beta_k\,|\xi|^{p-1}, \quad \text{for every } s \text{ such that } |s| \leq k, \tag{2.3}$$

$$\Big(a(x,t,s,\xi) - a(x,t,s,\overline{\xi}), \xi - \overline{\xi}\Big) > 0, \text{ if } \xi \neq \overline{\xi} \tag{2.4}$$

- $K : Q_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$\big|K(x,t,\eta)\big| \leq c(x,t)(\big|\eta\big|^\gamma + 1), \tag{2.5}$$

with

$$\gamma = \frac{N+2}{N+p}(p-1), \; c \in L^r(Q_T) \text{ and } r = \frac{N+p}{p-1}, \tag{2.6}$$

- $H : Q_T \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|H(x,t,\xi)| \leq b(x,t)(|\xi|^\delta + 1), \tag{2.7}$$

with

$$\delta = \frac{N(p-1)+p}{N+2} \text{ and } b \in L^{N+2,1}(Q_T), \tag{2.8}$$

for a.e. $(t,x) \in Q_T$, for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$. Moreover

$$f \in L^1(Q_T), \tag{2.9}$$

$$g \in \big(L^{p'}(Q_T)\big)^N \tag{2.10}$$

and

$$u_0 \in L^1(\Omega). \tag{2.11}$$

Under these assumptions, the above problem does not admit, in general, a solution in the sense of distribution since we cannot expect to have the fields $a(x,t,u,\nabla u)$, $K(x,t,u)$ in $(L^1_{loc}(Q_T))^N$ and $H(x,t,\nabla u)$ in $L^1_{loc}(Q_T)$. For this reason in the present paper we consider the framework of renormalized solutions.

For any $k > 0$ we denote by T_k the truncation function at height $\pm k$, $T_k = \max(-k, \min(k,s))$ for any $s \in \mathbb{R}$.

We use in the present paper the two Lorentz spaces $L^{q,1}(Q_T)$ and $L^{q,+\infty}(Q_T)$, see for example ([21, 24]) for references about Lorentz spaces $L^{q,s}$. If f^* denotes the decreasing rearrangement of a measurable function f,

$$f^*(r) = \inf\{s \geq 0 \,:\, \text{meas}\big\{(x,t) \in Q_T \,:\, |f(x,t)| > s\big\} < r\}, \quad r \in [0, \text{meas}(Q_T)],$$

$L^{q,1}(Q_T)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,1}(Q_T)} = \left(\int_0^{\text{meas}(Q_T)} f^*(r)\, r^{\frac{1}{q}} \frac{dr}{r}\right) < +\infty$$

while $L^{q,\infty}(Q_T)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,\infty}(Q_T)} = \sup_{r>0} r\left[\text{meas}\left\{(x,t)\in Q_T\ :\ |f(x,t)|>r\right\}\right]^{1/q} < +\infty.$$

If $1<q<+\infty$ we have the generalized Hölder inequality

$$\begin{cases} \forall f\in L^{q,\infty}(Q_T),\ \forall g\in L^{q',1}(Q_T), \\ \displaystyle\int_{Q_T}|fg| \le \|f\|_{L^{q,\infty}(Q_T)}\|g\|_{L^{q',1}(Q_T)}. \end{cases} \tag{2.12}$$

Following [3, 4] we recall the definition of a renormalized solution to Problem (2.1)

Definition 2.1. A real function u defined in Q_T is a renormalized solution of (2.1) if it satisfies the following conditions:

$$u\in L^\infty((0,T);L^1(\Omega)), \tag{2.13}$$

$$T_k(u)\in L^p((0,T);W_0^{1,p}(\Omega)),\ \text{for any}\ k>0, \tag{2.14}$$

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{(x,t)\in Q_T:|u(x,t)|\le n\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0, \tag{2.15}$$

and if for every function $S\in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\begin{aligned} \frac{\partial S(u)}{\partial t} &- \text{div}(a(x,t,u,\nabla u)S'(u)) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &+ H(x,t,\nabla u)S'(u) + \text{div}(K(x,t,u)S'(u)) - S''(u)K(x,t,u)\nabla u \\ &\qquad = fS'(u) - (\text{div}\, g)S'(u) \quad \text{in}\ \mathscr{D}'(Q) \end{aligned} \tag{2.16}$$

and

$$S(u)(t=0) = S(u_0) \quad \text{in}\ \Omega. \tag{2.17}$$

Remark 1. It is well known that conditions (2.13) and (2.14) allow to define ∇u almost everywhere in Q_T: for any $k>0$ we have $\nabla T_k(u) = \chi_{\{|u|<k\}}\nabla u$ a.e in Q_T where $\chi_{\{|u|<K\}}$ denotes the characteristic function of the set $\{(x,t);|u(x,t)|<k\}$. We notice that equation (2.16) can be formally obtained through pointwise multiplication of (2.1) by $S'(u)$ and all terms except $S(u)_t$ in (2.16) belong to $L^1(Q_T)+L^{p'}((0,T);W^{-1,p'}(\Omega))$ since $T_k(u)\in L^p((0,T);W_0^{1,p}(\Omega))$, for any $k>0$ and S' has a compact support. It follows that (2.16) has a meaning in $\mathscr{D}'(Q_T)$ and that the initial condition (2.17) makes sense. At last condition (2.15) gives additional information on ∇u for large value of $|u|$.

Remark 2. Growth assumptions (2.5) and (2.7) on K and H together with (2.6), (2.8), (2.13) and (2.15) allow to prove that

$$\lim_{n\to+\infty}\frac{1}{n}\int_{Q_T}|H(x,t,\nabla u)||T_n(u)|\,dx\,dt = 0, \tag{2.18}$$

$$\lim_{n\to+\infty}\frac{1}{n}\int_{Q_T}|K(x,t,u)||\nabla T_n(u)|\,dx\,dt = 0. \tag{2.19}$$

Indeed from (2.15) and the elliptic condition (2.2) together with (2.13) it follows that there exists $M, L > 0$ such that

$$\sup_{t\in(0,T)} \int_\Omega |T_n(u(t))|^2 dx + \int_{Q_T} |\nabla T_n(u)|^p dxdt \le Mn + L, \quad \forall n > 0,$$

and Lemma A.1 (see Appendix A) gives that $|\nabla u|^{\frac{N(p-1)+p}{N+2}} \in L^{\frac{N+2}{N+1},\infty}(Q_T)$. Therefore Hölder inequality, (2.7) and (2.8) imply that the field $H(x,t,\nabla u)$ belongs to $L^1(Q_T)$. Since u is finite almost everywhere, Lebesgue dominated convergence theorem yields that (2.18) holds true.

As far as (2.19) is concerned, assumption (2.5) leads to

$$\int_{Q_T} |K(x,t,u)||\nabla T_n(u)|dxdt \le \int_{Q_T} c(x,t)\,|u|^\gamma\,|\nabla T_n(u)|\,dxdt + \int_{Q_T} c(x,t)\,|\nabla T_n(u)|\,dxdt.$$

A few computations together with Hölder and Gagliardo-Nirenberg inequalities imply that

$$\begin{aligned}
&\int_{Q_T} c(x,t)\,|T_n(u)|^\gamma\,|\nabla T_n(u)|\,dxdt \\
&\le \left(\int_{Q_T} c^r(x,t)dxdt\right)^{\frac{1}{r}} \left(\int_{Q_T} |T_n(u)|^{\frac{(N+2)p}{N}}\,dxdt\right)^{\frac{N(p-1)}{p(N+p)}} \\
&\quad\times \left(\int_{Q_T} |\nabla T_n(u)|^p\,dxdt\right)^{\frac{1}{p}} \\
&\le n^{\frac{1}{r}} C\,\|c\|_{L^r(Q_T)}\,\|u\|^{\frac{1}{r}}_{L^\infty((0,T);L^1(\Omega))} \left(\int_{Q_T} |\nabla u|^p dxdt\right)^{\frac{N+1}{N+p}}.
\end{aligned}$$

Since $1-\frac{1}{r} = \frac{N+1}{N+p}$, the energy condition (2.15) and (2.2) implies that

$$\lim_{n\to+\infty} \frac{1}{n}\int_{Q_T} c(x,t)\,|T_n(u)|^\gamma\,|\nabla T_n(u)|\,dxdt = 0.$$

Similar arguments show that

$$\lim_{n\to+\infty} \frac{1}{n}\int_{Q_T} c(x,t)\,|\nabla T_n(u)|\,dxdt = 0,$$

so that (2.19) holds.

Notation. In the whole paper, for any measurable function v defined on Q_T and any $k > 0$, we denote by $\{|v| \le k\}$ (respectively $\{|v| < k\}$) the measurable subset $\{(x,t) \in Q_T : |v(x,t)| \le k\}$ (respectively $\{(x,t) \in Q_T : |v(x,t)| < k\}$.

3. Existence result

The main result of the present paper is the following existence result.

Theorem 3.1. *Under the assumptions (2.2)-(2.11) there exists at least a renormalized solution to Problem (2.1).*

Remark 3 (Comparison with the elliptic version of (1.1))**.** The nonlinear and noncoercive elliptic equation

$$-\Delta_p u + \operatorname{div}(c|u|^{\gamma-1}u) + b|\nabla u|^{\delta} = f - \operatorname{div} g,$$

is studied for example in [14, 18, 19] and the main conditions on the exponent are $\gamma \le p-1$ and $\delta \le p-1$. It is worth noting (see Figure 1) that for $1 < p \le 2$ we have

$$\frac{N+2}{N+p}(p-1) \ge p-1 \quad \text{and} \quad \frac{N(p-1)+p}{N+2} \ge p-1$$

while for $p \ge 2$

$$\frac{N+2}{N+p}(p-1) \le p-1 \quad \text{and} \quad \frac{N(p-1)+p}{N+2} \le p-1.$$

This difference is mainly due to the presence of the time derivative of u in the parabolic equation which modifies the control of u with respect to p.

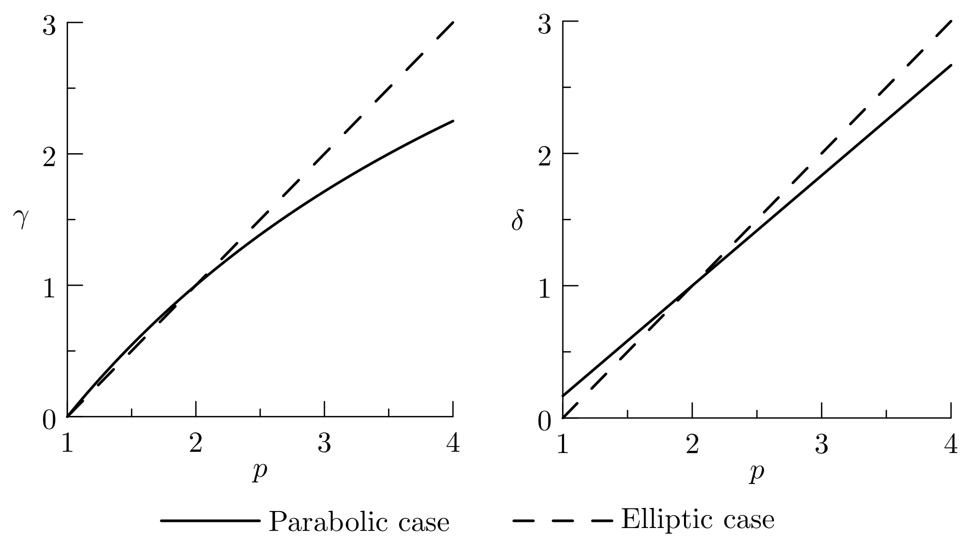

FIGURE 1

Roughly speaking in the elliptic case the estimate

$$\int_\Omega |\nabla T_k(u)|^p \, dx \le kM + L, \quad \forall k > 0$$

implies that $\||u|^{p-1}\|_{L^q(\Omega)} \le CM$ (with $q < N/(N-p)$) while in the parabolic case the estimate

$$\sup_{t\in(0,T)} \int_\Omega |T_k(u(t))|^2 + \int_0^T \int_\Omega |\nabla T_k(u)|^p \le kM + L$$

leads to a control of $|u|^{(N(p-1)+p)/(N+p)}$ with respect to M and L (see Lemma A.1 in Appendix A).

Moreover it worth noting that in the elliptic case when $\gamma = \delta = p-1$ a smallness condition on b or c –in an appropriate Lesbesgue or Lorentz space– seems to be necessary to obtain the existence of a solution. In our existence result we do not need such a smallness condition.

Proof of Theorem 3.1. In Step 1 we define u_ε solution of an approximate problem. Step 2 is devoted to obtain a priori estimates. In Step 3 we obtain some convergence results and we conclude the proof in Step 4 by passing to the limit in the approximate problem.

Step 1 (Approximate problem). For $\varepsilon > 0$ let us consider the following approximated problem

$$(3.1)\qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) + \operatorname{div} K_\varepsilon(x,t,u_\varepsilon) \\ \qquad\qquad + H_\varepsilon(x,t,\nabla u_\varepsilon) = f_\varepsilon - \operatorname{div} g \quad \text{in } Q_T \\ \qquad\qquad u_\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \qquad\qquad u_\varepsilon(x,0) = (u_0)_\varepsilon(x) \quad \text{in } \Omega \end{cases}$$

where

$$(3.2)\qquad a_\varepsilon(x,t,s,\xi) = a(x,t,T_{\frac{1}{\varepsilon}}(s),\xi),$$

$$(3.3)\qquad f_\varepsilon \in L^{p'}(Q_T) \quad \text{and} \quad f_\varepsilon \to f \text{ in } L^1(Q_T) \text{ and a.e. in } Q_T,$$

$$(3.4)\qquad (u_0)_\varepsilon \in L^2(\Omega) \quad (u_0)_\varepsilon \to u_0 \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega,$$

$$(3.5)\qquad K_\varepsilon(x,t,\eta) = K(x,t,T_{\frac{1}{\varepsilon}}(\eta)),$$

$$(3.6)\qquad \begin{cases} \left|K_\varepsilon(x,t,\eta)\right| \le \left|K(x,t,\eta)\right| \le c(x,t)(\left|\eta\right|^\gamma + 1) \\ \left|K_\varepsilon(x,t,\eta)\right| \le c(x,t)\left(\left(\frac{1}{\varepsilon}\right)^\gamma + 1\right). \end{cases}$$

and

$$(3.7)\qquad H_\varepsilon(x,t,\xi) = T_{\frac{1}{\varepsilon}}\left(H(x,t,\xi)\right),$$

$$(3.8)\qquad \begin{cases} |H_\varepsilon(x,t,\xi)| \le |H(x,t,\xi)| \le b(x,t)(|\xi|^\gamma + 1) \\ |H_\varepsilon(x,t,\xi)| \le \frac{1}{\varepsilon}, \end{cases}$$

By known results there exists at least a weak solution, u_ε to (3.1) which belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ (see [20]).

Step 2 (A priori estimates) Let $k > 0$. If we take $T_k(u_\varepsilon)$ as test function in (3.1) and we integrate between $(0,t)$ for almost any $t \in (0,T)$, using (3.6) and (3.8) we have

$$\begin{aligned}
&\int_0^t < \frac{\partial u_\varepsilon}{\partial t}, T_k(u_\varepsilon) > dt + \int_0^t \int_\Omega a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon)\,dxdt \\
&\le \int_0^t \int_\Omega c(x,t)\,|u_\varepsilon|^\gamma\,|\nabla T_k(u_\varepsilon)|\,dxdt + \int_0^t \int_\Omega c(x,t)\,|\nabla T_k(u_\varepsilon)|\,dxdt \\
&\quad + \int_0^t \int_\Omega b(x,t)\,|T_k(u_\varepsilon)|\,|\nabla u_\varepsilon|^\delta\,dxdt + \int_0^t \int_\Omega b(x,t)\,|T_k(u_\varepsilon)|\,dxdt \\
&\quad + \int_0^t \int_\Omega f_\varepsilon T_k(u_\varepsilon)\,dxdt + \int_0^t \int_\Omega g\nabla T_k(u_\varepsilon)\,dxdt,
\end{aligned} \tag{3.9}$$

where $<,>$ denotes the duality bracket between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. If we define the function Ψ_k by

$$\Psi_k(s) = \int_0^s T_k(\tau)\,d\tau, \tag{3.10}$$

we have for almost any $t \in (0,T)$ that

$$\int_0^t < \frac{\partial u_\varepsilon}{\partial t}, T_k(u_\varepsilon) > dt = \int_\Omega \Psi_k(u_\varepsilon(t))\,dx - \int_\Omega \Psi_k((u_0)_\varepsilon)\,dx.$$

Since

$$\frac{1}{2}|T_k(s)|^2 \le \frac{1}{2} s T_k(s) \le \Psi_k(s) \le k|s| \quad \forall s \in \mathbb{R}, \tag{3.11}$$

we obtain that

$$\int_0^t < \frac{\partial u_\varepsilon}{\partial t}, T_k(u_\varepsilon) > dt \ge \frac{1}{2}\int_\Omega u_\varepsilon(t) T_k(u_\varepsilon(t))\,dx - k\int_\Omega |(u_0)_\varepsilon|, \tag{3.12}$$

for almost any $t \in (0,T)$. Using (2.2) and (3.12) we deduce from (3.9)

$$\begin{aligned}
&\frac{1}{2}\int_\Omega u_\varepsilon(t) T_k(u_\varepsilon(t))\,dx + \alpha \int_0^t \int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt \\
&\le \int_0^t \int_\Omega c(x,t)\,|u_\varepsilon|^\gamma\,|\nabla T_k(u_\varepsilon)|\,dxdt + \int_0^t \int_\Omega c(x,t)\,|\nabla T_k(u_\varepsilon)|\,dxdt \\
&\quad + k\int_0^t \int_\Omega b(x,t)\,|\nabla u_\varepsilon|^\delta\,dxdt + k\int_0^t \int_\Omega b(x,t)\,dxdt + k\int_0^t \int_\Omega \left|f_\varepsilon\right|\,dxdt \\
&\quad + k\int_\Omega |(u_0)_\varepsilon|\,dx + \int_0^t \int_\Omega g\nabla T_k(u_\varepsilon)\,dxdt,
\end{aligned}$$

for almost any $t \in (0,T)$. Using Young inequality we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega u_\varepsilon(t)T_k(u_\varepsilon(t))\,dx + \alpha\int_0^t\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dx\,dt \\
&\le \int_0^t\int_\Omega c(x,t)\,|u_\varepsilon|^\gamma\,|\nabla T_k(u_\varepsilon)|\,dx\,dt + k\int_0^t\int_\Omega b(x,t)\,|\nabla u_\varepsilon|^\delta\,dx\,dt \\
&\quad + k\int_0^t\int_\Omega b(x,t)\,dx\,dt + k\int_0^t\int_\Omega \left|f_\varepsilon\right|\,dx\,dt + k\int_\Omega |(u_0)_\varepsilon|\,dx\,dt \\
&\quad + \frac{\alpha}{p}\int_0^t\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dx\,dt + \frac{\left(\frac{\alpha}{2}\right)^{-\frac{p'}{p}}}{p'}\left(\|c\|^{p'}_{L^{p'}(Q_T)} + \left\|g\right\|^{p'}_{L^{p'}(Q_T)}\right).
\end{aligned}$$

If we take the supremum for $t \in (0,t_1)$, where $t_1 \in (0,T)$ will be chosen later, we have

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega u_\varepsilon(t)T_k(u_\varepsilon(t))\,dx + \frac{\alpha}{p'}\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dx\,dt \\
&\le \int_0^{t_1}\int_\Omega c(x,t)\,|u_\varepsilon|^\gamma\,|\nabla T_k(u_\varepsilon)|\,dx\,dt + k\int_0^{t_1}\int_\Omega b(x,t)\,|\nabla u_\varepsilon|^\delta\,dx\,dt \\
&\quad + k\int_0^{t_1}\int_\Omega b(x,t)\,dx\,dt + k\int_0^{t_1}\int_\Omega \left|f_\varepsilon\right|\,dx\,dt + k\int_\Omega |(u_0)_\varepsilon|\,dx \\
&\quad + \frac{\left(\frac{\alpha}{2}\right)^{-\frac{p'}{p}}}{p'}\left(\|c\|^{p'}_{L^{p'}(Q_T)} + \left\|g\right\|^{p'}_{L^{p'}(Q_T)}\right).
\end{aligned} \tag{3.13}$$

Now we estimate $\int_0^{t_1}\int_\Omega c(x,t)\,|u_\varepsilon|^\gamma\,|\nabla T_k(u_\varepsilon)|\,dxdt$. Using Hölder inequality, Gagliardo-Nirenberg inequality together with Young inequality yields that

$$
\begin{aligned}
(3.14)\qquad &\int_0^{t_1}\int_\Omega c(x,t)\,|T_k(u_\varepsilon)|^\gamma\,|\nabla T_k(u_\varepsilon)|\,dxdt\\
&\le \left(\int_0^{t_1}\int_\Omega c^{p'}(x,t)\,|T_k(u_\varepsilon)|^{\gamma p'}\,dxdt\right)^{\frac{1}{p'}}\left(\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\right)^{\frac{1}{p}}\\
&\le \left(\int_0^{t_1}\int_\Omega c^{r}(x,t)dxdt\right)^{\frac{1}{r}}\left(\int_0^{t_1}\int_\Omega |T_k(u_\varepsilon)|^{\frac{(N+2)p}{N}}\,dxdt\right)^{\frac{N(p-1)}{p(N+p)}}\\
&\quad\times\left(\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\right)^{\frac{1}{p}}\\
&\le C_1\,\|c\|_{L^r(Q_{t_1})}\left(\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon(t))|^2\,dx\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\right)^{\frac{N+1}{N+p}}\\
&\le C_1\,\|c\|_{L^r(Q_{t_1})}\left[\frac{1}{r}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon(t))|^2\,dx\right.\\
&\quad\left.+\frac{N+1}{N+p}\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\right],
\end{aligned}
$$

where C_1 is a constant that depends only on N and p.

Using (3.14) together with Hölder inequality in (3.13) we get

$$
\begin{aligned}
(3.15)\qquad &\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega u_\varepsilon(t)\,T_k(u_\varepsilon(t))\,dx+\frac{\alpha}{p'}\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\\
&\le C_1\,\|c\|_{L^r(Q_{t_1})}\left[\frac{1}{r}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon(t))|^2\,dx\right.\\
&\quad\left.+\frac{N+1}{N+p}\int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\right]+kM+L,
\end{aligned}
$$

where

$$M=M^*+\left\||\nabla u_\varepsilon|^\delta\right\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})}\|b\|_{L^{N+2,1}(Q_{t_1})}$$

with

$$M^*=\|f\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}+\|b\|_{L^{N+2,1}(Q_T)},$$

and

$$L=\frac{\left(\frac{\alpha}{2}\right)^{-\frac{p'}{p}}}{p'}\left(\|c\|^{p'}_{L^{p'}(Q_T)}+\|g\|_{L^{p'}(Q_T)}\right).$$

Since

$$\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon(t))|^2dx\le\sup_{t\in(0,t_1)}\int_\Omega u_\varepsilon(t)\,T_k(u_\varepsilon(t))\,dx,$$

if t_1 verifies

$$\left(\frac{1}{2} - C_1 \|c\|_{L^r(Q_{t_1})} \frac{1}{r}\right) > 0 \text{ and } \left(\frac{\alpha}{p'} - C_1 \|c\|_{L^r(Q_{t_1})} \frac{N+1}{N+p}\right) > 0 \tag{3.16}$$

then (3.15) leads to

$$C_2\left(\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_\varepsilon(t))|^2\,dx + \int_0^{t_1}\int_\Omega |\nabla T_k(u_\varepsilon)|^p\,dxdt\right) \le L + Mk,$$

for any $k > 0$, where

$$C_2 = \min\left\{\frac{1}{2} - C_1\|c\|_{L^r(Q_{t_1})}\frac{1}{r}, \frac{\alpha}{p'} - C_1\|c\|_{L^{p'\frac{N+2}{N}}(Q_{t_1})}\frac{N+1}{N+p}\right\}.$$

Using Lemma A.1 (see Appendix A) we obtain

$$\begin{aligned}\left\||\nabla u_\varepsilon|^\delta\right\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})} &= \left\||\nabla u_\varepsilon|^{p-1}\right\|^{\delta\frac{1}{p-1}}_{L^{\frac{(N+1)p-N}{(N+1)(p-1)},\infty}(Q_{t_1})} \\ &\le C_2\left(M^* + \left\||\nabla u_\varepsilon|^\delta\right\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})}\|b\|_{L^{N+2,1}(Q_{t_1})}\right) \\ &\quad + C_2\left|Q_{t_1}\right|^{\frac{N}{(N+2)p}} L^{\frac{N(p-1)+p}{(N+2)p}}\end{aligned} \tag{3.17}$$

where $C_2 = C_2(N, p, \|c\|, \alpha)$. If we choose t_1 such that (3.16) hold and

$$1 - C_2\|b\|_{L^{N+2,1}(Q_{t_1})} > 0 \tag{3.18}$$

from (3.17) it follows that

$$\left\||\nabla u_\varepsilon|^\delta\right\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})} \le C_3, \tag{3.19}$$

where $C_3 = C_3(N, p, \alpha, \left|Q_{t_1}\right|, \|c\|, \|b\|, \|u_0\|, \|f\|, \|g\|)$.
Now we are able to prove the a priori estimate for u_ε.
Let us turn back to (3.15). Using Lemma A.1 and Hölder inequality we have

$$\begin{aligned}\left\||u_\varepsilon|^{\frac{N(p-1)+p}{N+p}}\right\|_{L^{\frac{N+p}{N},\infty}(Q_{t_1})} \le& C_4\left[M^* + \left\||\nabla u_\varepsilon|^\delta\right\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})}\|b\|_{L^{N+2,1}(Q_{t_1})}\right] \\ &+ C_4\left|Q_{t_1}\right|^{\frac{Np}{(N+2)}} L^{\frac{N(p-1)+p}{(N+2)p}}.\end{aligned}$$

Using (3.19) we have

$$\left\||u_\varepsilon|^{\frac{N(p-1)+p}{N+p}}\right\|_{L^{\frac{N+p}{N},\infty}(Q_{t_1})} \le C_5 \tag{3.20}$$

where $C_5 = C_5(N, p, \alpha, \left|Q_{t_1}\right|, \|c\|, \|b\|, \|u_0\|, \|f\|, \|g\|)$.

Since

$$\|u_\varepsilon\|_{L^\infty(0,t_1;L^1(\Omega))} \le \text{meas}(\Omega) + \sup_{t\in(0,t_1)}\int_\Omega u_\varepsilon(t)T_1(u_\varepsilon(t))\,dx$$

from (3.15)–(3.19) it follows that

$$\|u_\varepsilon\|_{L^\infty(0,t_1;L^1(\Omega))} \le C_6, \tag{3.21}$$

where $C_6 = C_6(N, p, \alpha, \left|Q_{t_1}\right|, \|c\|, \|b\|, \|u_0\|, \|f\|, \|g\|)$.

Now we use the same technique as in [26]. We consider a partition of the entire interval $[0,T]$ into a finite number of intervals $[0,t_1]$, $[t_1,t_2]$, ..., $\left[t_{n-1,}T\right]$ such that for each interval $[t_{i-1},t_i]$ a similar condition to (3.16) and (3.18) holds. In this way in each cylinder $Q_{t_i}=\Omega\times[t_{i-1},t_i]$ we obtain a priori estimates of type (3.19), (3.20) and (3.21). Then we can deduce that

$$\left\| |\nabla u_\varepsilon|^\delta \right\|_{L^{\frac{N+2}{N+1},\infty}(Q_T)} \le C_7, \tag{3.22}$$

$$\left\| |u_\varepsilon|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q_T)} \le C_8, \tag{3.23}$$

$$\| |u_\varepsilon| \|_{L^\infty(0,T;L^1(\Omega))} \le C_9 \tag{3.24}$$

for some constants C_7, C_8 and C_9 depending on N, p, α, $|Q_T|$, $\|c\|$, $\|b\|$, $\|u_0\|$, $\|f\|$ and $\|g\|$.

As a consequence of (3.23), Hölder inequality implies that

$$b(x,t)|\nabla u_\varepsilon|^\delta \text{ is bounded in } L^1(Q_T)$$

and then inequality (3.13) with T in place of t_1 allows us to prove that

$$T_k(u_\varepsilon) \text{ is bounded in } L^p((0,T);W_0^{1,p}(\Omega)), \tag{3.25}$$

independently of ε for any $k\ge 0$.

Step 3. Proceeding as in [3] and [6], it is possible to prove that for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' is compact the term

$$\frac{\partial S(u_\varepsilon)}{\partial t} \text{ is bounded in } L^1(Q_T)+L^{p'}((0,T);W^{-1,p'}(\Omega)), \tag{3.26}$$

independently of ε. Indeed, by pointwise multiplication of $S'(u_\varepsilon)$ in (3.1) we have

$$\begin{aligned}\frac{\partial S(u_\varepsilon)}{\partial t} &- \operatorname{div}(a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)S'(u_\varepsilon))\\ &+ S''(u_\varepsilon)a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon + H_\varepsilon(x,t,\nabla u_\varepsilon)S'(u_\varepsilon)\\ &+ \operatorname{div}(K_\varepsilon(x,t,u_\varepsilon)S'(u_\varepsilon)) - S''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon\\ &= f_\varepsilon S'(u_\varepsilon) - \operatorname{div}\big(gS'(u_\varepsilon)\big) + S''(u_\varepsilon)g\nabla u_\varepsilon \quad in\ \mathscr{D}'(\Omega).\end{aligned} \tag{3.27}$$

Now each term in (3.27) is estimated as follows. Recalling that $S'(u_\varepsilon)$ has a compact support contained in $[-k,k]$, because of (2.3), (3.2) and (3.25) the term

$$\begin{aligned}\operatorname{div}(a(x,t,u_\varepsilon,\nabla u_\varepsilon)S'(u_\varepsilon)) - S''(u_\varepsilon)a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon&\\ + f_\varepsilon S'(u_\varepsilon) - \operatorname{div}\big(gS'(u_\varepsilon)\big) + S''(u_\varepsilon)g\nabla u_\varepsilon&\end{aligned}$$

is bounded in $L^1(Q_T)+L^{p'}((0,T);W^{-1,p'}(\Omega))$ independently of ε.

From (3.5) and (3.25) it follows that for $0 < \varepsilon < \frac{1}{k}$

$$\begin{aligned}\int_0^t \int_\Omega \left|K_\varepsilon(x,t,u_\varepsilon)S'(u_\varepsilon)\right|^{p'} dxdt \le& \int_0^t \int_\Omega c^{p'} \ |T_k(u_\varepsilon)|^{p'\gamma} \left|S'(u_\varepsilon)\right|^{p'} dxdt \\ &+ \int_0^t \int_\Omega c(x,t)^{p'} \left|S'(u_\varepsilon)\right|^{p'} dxdt \\ \le& \left\|S'\right\|_\infty^{p'} k^{p'\gamma} \|c\|^{p'}_{L^{p'}(Q_T)} + \left\|S'\right\|_\infty \ \|c\|^{p'}_{L^{p'}(Q_T)} \\ \le& C_{10},\end{aligned}$$

$$\begin{aligned}\int_0^T \int_\Omega \left|S''(u_\varepsilon)K_\varepsilon(x,t,u_\varepsilon)\nabla u_\varepsilon\right| \\ \le \left\|S''\right\|_\infty \ \|c\|_{L^{p'}(Q_T)} \left\||\nabla T_k(u_\varepsilon)|\right\|_{L^p(Q_T)} (k^\gamma + 1) \le C_{11},\end{aligned} \tag{3.28}$$

for some constants C_{10} and C_{11} independently on ε. Similarly by (3.7) and (3.25) we have

$$\begin{aligned}&\int_0^t \int_\Omega |S'(u_\varepsilon)H_\varepsilon(x,t,\nabla T_k(u_\varepsilon))|dxdt \\ &\le \int_0^t \int_\Omega S'(u_\varepsilon)b(x,t)\,|\nabla u_\varepsilon|^\delta \,dxdt + \int_0^t \int_\Omega S'(u_\varepsilon)b(x,t)dxdt \\ &\le \left\|S'\right\|_\infty \left(\|b\|_{L^{\frac{p(N+2)}{N+p}}(Q_T)} \left\||\nabla T_k(u_\varepsilon)|\right\|^\delta_{L^p(Q_T)} + \|b\|_{L^1(Q_T)}\right) \\ &\le C_{12}\end{aligned}$$

for some constant C_{12} independently on ε. All the previous estimates prove (3.26).

Following [3, 6], estimates (3.25) and (3.26) together with Aubin type lemma (see [30]) imply that there exists a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \to u \text{ a.e. in } Q_T, \tag{3.29}$$

where u is a measurable function defined on Q_T.

Due to (3.25) and (2.3) there exists a subsequence of u_ε, still indexed by ε, such that

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } L^p((0,T);W_0^{1,p}(\Omega)), \tag{3.30}$$

$$a_\varepsilon(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon) \rightharpoonup \omega_k \text{ weakly in } (L^{p'}(Q_T))^N \tag{3.31}$$

as ε goes to zero, for any $k \ge 0$ and where for any $k \ge 0$ the field ω_k belongs to $(L^{p'}(Q_T))^N$.

Dunford-Pettis theorem allows us to show that the sequence $H_\varepsilon(x,t,\nabla u_\varepsilon)$ is weakly compact in $L^1(Q_T)$. Indeed if E is a measurable set of Q_T, due to growth assumption (2.7) on H, estimate (3.22) yields that

$$\begin{aligned}\int_E |H_\varepsilon(x,t,\nabla u_\varepsilon)|dxdt \le& \int_E b(x,t)\left(|\nabla u_\varepsilon|^\gamma + 1\right)dxdt \\ \le& \|b\|_{L^{N+2,1}(E)} C_7 + \|b\|_{L^1(E)}.\end{aligned}$$

Since b belongs to $L^{N+2,1}(Q_T)$, the sequence $H_\varepsilon(x,t,\nabla u_\varepsilon)$ is equi-integrable in $L^1(Q_T)$. It follows that there exists Λ belonging to $L^1(Q_T)$ and a subsequence of $H_\varepsilon(x,t,\nabla u_\varepsilon)$, still indexed by ε, such that

$$H_\varepsilon(x,t,\nabla u_\varepsilon) \rightharpoonup \Lambda \text{ weakly in } L^1(Q_T) \tag{3.32}$$

as ε goes to zero.

We are now in a position to prove that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int\limits_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dxdt = 0. \tag{3.33}$$

Using the admissible test function $\frac{T_n(u_\varepsilon)}{n}$ in (3.1) and recalling (3.11) yield that

$$\begin{aligned}
&\frac{1}{2n}\sup_{t\in(0,T)}\int_\Omega |T_n(u_\varepsilon(t))|^2\,dx + \frac{1}{n}\int\limits_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dxdt\\
&\le \frac{1}{n}\int_0^T\int_\Omega |K_\varepsilon(x,t,u_\varepsilon)||\nabla T_n(u_\varepsilon)|\,dxdt\\
&\quad-\frac{1}{n}\int_0^T\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\int_\Omega \Psi_n((u_0)_\varepsilon)\,dx\\
&\quad+\frac{1}{n}\int_0^T\int_\Omega f_\varepsilon T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\int_0^T\int_\Omega g|\nabla T_n(u_\varepsilon)|\,dxdt.
\end{aligned} \tag{3.34}$$

By (3.6) for $0<\varepsilon<\frac{1}{n}$ we have

$$\begin{aligned}
&\frac{1}{n}\int\limits_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dxdt\\
&\le \frac{1}{n}\int_0^T\int_\Omega c|T_n(u_\varepsilon)|^\gamma|\nabla T_n(u_\varepsilon)|\,dxdt + \frac{1}{n}\int_0^T\int_\Omega c|\nabla T_n(u_\varepsilon)|\,dxdt\\
&\quad-\frac{1}{n}\int_0^T\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\int_\Omega \Psi_n((u_0)_\varepsilon)\,dx\\
&\quad+\frac{1}{n}\int_0^T\int_\Omega f_\varepsilon T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\int_0^T\int_\Omega g|\nabla T_n(u_\varepsilon)|\,dxdt.
\end{aligned}$$

Using Young inequality and the elliptic condition (2.2) on the operator a leads to

$$\begin{aligned}
&\frac{1}{2n}\int\limits_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dxdt\\
&\le \frac{1}{n}\int_0^T\int_\Omega c|T_n(u_\varepsilon)|^\gamma|\nabla T_n(u_\varepsilon)|\,dxdt + \frac{1}{n}\left(\frac{4}{\alpha}\right)^{p'}\|c\|^{p'}_{L^{p'}(Q_T}\\
&\quad-\frac{1}{n}\int_0^T\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\int_\Omega \Psi_n((u_0)_\varepsilon)\,dx\\
&\quad+\frac{1}{n}\int_0^T\int_\Omega f_\varepsilon T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\left(\frac{4}{\alpha}\right)^{p'}\|g\|^{p'}_{L^{p'}(Q_T}
\end{aligned} \tag{3.35}$$

Now we want to pass to the limit in each term in (3.35) as $\varepsilon \to 0$ and $n \to +\infty$. Let be $R > 0$ (where R will be chosen later). If we denote by

$$E_R = \left\{(x,t) \in Q_T : |u_\varepsilon(x,t)| > R\right\},$$

we have for $n > R$

$$\begin{aligned}(3.36)\quad & \frac{1}{n}\int_0^T\int_\Omega c\,|T_n(u_\varepsilon)|^\gamma\,|\nabla T_n(u_\varepsilon)|\,dxdt \\ &= \frac{1}{n}\int_{Q_T\setminus E_R} c\,|T_R(u_\varepsilon)|^\gamma\,|\nabla T_R(u_\varepsilon)|\,dxdt + \frac{1}{n}\int_{E_R} c\,|T_n(u_\varepsilon)|^\gamma\,|\nabla T_n(u_\varepsilon)|\,dxdt.\end{aligned}$$

Furthermore, by Gagliardo-Nirenberg inequality and by Young inequality and recalling that u_ε is bounded in $L^\infty((0,T);L^1(\Omega))$ we have

$$\begin{aligned}(3.37)\quad & \int_{E_R} c\,|T_n(u_\varepsilon)|^\gamma\,|\nabla T_n(u_\varepsilon)|\,dxdt \\ &\le C_1\,\|c\|_{L^r(E_R)}\left(\sup_{t\in(0,T)}\int_\Omega |T_n(u_\varepsilon(t))|^2\,dx\right)^{\frac{1}{r}} \\ &\quad\times\left(\int_0^T\int_\Omega |\nabla T_n(u_\varepsilon)|^p\,dxdt\right)^{\frac{N+1}{N+p}} \\ &\le C_1\,\|c\|_{L^r(E_R)}\,n^{\frac{1}{r}}\left(\sup_{t\in(0,T)}\int_\Omega |T_n(u_\varepsilon(t))|\,dx\right)^{\frac{1}{r}} \\ &\quad\times\left(\int_0^T\int_\Omega |\nabla T_n(u_\varepsilon)|\,dxdt^p\right)^{\frac{N+1}{N+p}} \\ &\le C_{13}\,\|c\|^{\frac{1}{r}}_{L^r(E_R)}\,n + \frac{\alpha}{4}\int_0^T\int_\Omega |\nabla T_n(u_\varepsilon)|^p\,dxdt,\end{aligned}$$

where C_{13} is independent on ε and R. Inserting (3.36) and (3.37) in (3.35) and using again the elliptic condition on a we obtain

$$\begin{aligned}&\frac{1}{4n}\int_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dxdt \\ &\le \frac{1}{n}\int_{Q_T\setminus E_R} c\,|T_R(u_\varepsilon)|^\gamma\,|\nabla T_R(u_\varepsilon)|\,dxdt + C_{13}\,\|c\|^{\frac{1}{r}}_{L^r(E_R)} \\ &\quad+\frac{1}{n}\int_0^T\int_\Omega |H_\varepsilon(x,t,\nabla u_\varepsilon)|\,|T_n(u_\varepsilon)|\,dxdt + \frac{1}{n}\int_\Omega \Psi_n((u_0)_\varepsilon)\,dx \\ &\quad+\frac{1}{n}\int_0^T\int_\Omega f_\varepsilon T_n(u_\varepsilon)\,dxdt + \frac{1}{n}\Big(\frac{4}{\alpha}\Big)^{p'}\Big(\|g\|^{p'}_{L^{p'}(Q_T} + \|c\|^{p'}_{L^{p'}(Q_T}\Big).\end{aligned}$$

Since $T_R(u_\varepsilon) \in L^p((0,T);W_0^{1,p}(\Omega))$ it follows that

$$(3.38)\quad \lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{Q_T\setminus E_R} c\,|T_R(u_\varepsilon)|^\gamma\,|\nabla T_R(u_\varepsilon)|\,dxdt = 0, \qquad \forall R > 0.$$

Since u_ε converges to u almost everywhere while $|T_n(s)| \le n$, Lebesgue dominated convergence theorem implies that $T_n(u_\varepsilon)$ converges to $T_n(u)$ in $L^\infty(Q_T)$ weak-$*$. The weak convergence of $H_\varepsilon(x,t,\nabla u_\varepsilon)$ in $L^1(Q_T)$ and Egorov theorem allow us to prove that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)T_n(u_\varepsilon)\,dx\,dt = \int_0^T\int_\Omega \Lambda T_n(u)\,dx\,dt. \tag{3.39}$$

Since u is finite almost everywhere the field $T_n(u)/n$ goes to zero almost everywhere in Q_T. From Lebesgue dominated convergence theorem it follows that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\frac{1}{n}\int_0^T\int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)T_n(u_\varepsilon)\,dx\,dt = 0 \tag{3.40}$$

Similar arguments lead to

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\frac{1}{n}\int_0^T\int_\Omega f_\varepsilon T_n(u_\varepsilon)\,dx\,dt = 0, \tag{3.41}$$

$$\frac{1}{n}\int_\Omega \Psi_n((u_0)_\varepsilon)\,dx = 0 \tag{3.42}$$

while

$$\lim_{n\to+\infty}\frac{1}{n}\left(\frac{4}{\alpha}\right)^{p'}\left(\|g\|^{p'}_{L^{p'}(Q_T}+\|c\|^{p'}_{L^{p'}(Q_T}\right) = 0 \tag{3.43}$$

Gathering (3.38)-(3.43) we get

$$\limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int\limits_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dx\,dt \le C_{14}(\|c\|_{L^r(E_R)}), \tag{3.44}$$

for any $R>0$. Finally, since u is finite almost everywhere in Q_T, $c\in(L^r(Q_T))^N$ yields that

$$\lim_{R\to+\infty}\|c\|_{L^r(E_R)} = 0$$

and it allows us to obtain (3.33).

Step 4. We now pass to the limit in the approximate problem.

Since $H_\varepsilon(x,t,\nabla u_\varepsilon)$ converges to Λ weakly in $L^1(Q_T)$ the method developed in [4, 5] allows us to pass to the limit (3.27). We omit the proof in the present paper. Observe that Egorov theorem helps in dealing with the field $H_\varepsilon(x,t,\nabla u_\varepsilon)$ and the presence of the term $\operatorname{div}(K_\varepsilon(x,t,u_\varepsilon))$ is studied in [15]. It follows that u is a renormalized solution to

$$\begin{cases} u_t - \operatorname{div}(a(x,t,u,\nabla u)) & \\ \qquad + \operatorname{div}(K(x,t,u)) + \Lambda = f - \operatorname{div} g & \text{in } Q_T\\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T)\\ u(x,0) = u_0(x) & \text{in } \Omega,\end{cases}$$

and that

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \tag{3.45}$$

weakly in $L^1(Q_T)$ as ε goes to zero, for any $k>0$.

To conclude the proof it remains to show that $\Lambda = H(x,t,\nabla u)$. Since the operator a is strictly monotone (see assumption (2.4)), it is well know that (3.45) implies that for any $k > 0$ $T_k(u_\varepsilon)$ strongly converges $T_k(u)$ in $L^p((0,T);W_0^{1,p}(\Omega))$ (see Lemma 5 in [9]) Recalling that u is finite almost everywhere, we deduce that up to a subsequence ∇u_ε converges to ∇u almost everywhere in Q_T. Recalling that Λ is the weak limit of $H_\varepsilon(x,t,\nabla u_\varepsilon)$ we conclude that $\Lambda = H(x,t,\nabla u)$.

The proof of Theorem 3.1 is now complete. □

Remark 4. Let us consider the following nonlinear problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a(x,t,u_\varepsilon,\nabla u_\varepsilon) + \operatorname{div} K_\varepsilon(x,t,u_\varepsilon) \\ \qquad\qquad + H_\varepsilon(x,t,\nabla u_\varepsilon) = f_\varepsilon - \operatorname{div} g_\varepsilon \quad \text{in } Q_T \\ \qquad\qquad u_\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \qquad\qquad u_\varepsilon(x,0) = (u_0)_\varepsilon(x) \quad \text{in } \Omega \end{cases} \tag{3.46}$$

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $K_\varepsilon : Q_T \times \mathbb{R} \to \mathbb{R}^N$, $H_\varepsilon : Q_T \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that (2.2)-(2.4) hold and

$$\left|K_\varepsilon(x,t,\eta)\right| \le c(x,t)(\left|\eta\right|^\gamma + 1),$$

$$|H_\varepsilon(x,t,\xi)| \le b(x,t)(|\xi|^\delta + 1).$$

Let us denote by $K : Q_T \times \mathbb{R} \to \mathbb{R}^N$, $H : Q_T \times \mathbb{R}^N \to \mathbb{R}$ two Carathéodory functions and assume that

$$\begin{cases} K_\varepsilon(x,t,s_\varepsilon) \to K(x,t,s) \\ \text{for every sequence } s_\varepsilon \in \mathbb{R} \text{ such that} \\ s_\varepsilon \to s \text{ a.e in } Q_T, \end{cases}$$

$$\begin{cases} H_\varepsilon(x,t,\eta_\varepsilon) \to H(x,t,\eta) \\ \text{for every sequence } \eta_\varepsilon \in \mathbb{R}^N \text{ such that} \\ \eta_\varepsilon \to \eta \text{ a.e in } Q_T, \end{cases}$$

then K and H verify (2.5) and (2.7). Moreover let be f_ε, $(u_0)_\varepsilon$, g_ε sequences in $L^1(Q_T)$, $L^1(\Omega)$, $(L^{p'}(Q_T))^N$ such that

$$f_\varepsilon \to f \text{ strongly in } L^1(Q_T) \tag{3.47}$$

$$(u_0)_\varepsilon \to u_0 \text{ strongly in } L^1(\Omega) \tag{3.48}$$

$$g_\varepsilon \to g \text{ strongly in } (L^{p'}(Q_T))^N. \tag{3.49}$$

From Theorem 3.1 let u_ε be a renormalized solution of (3.46). We are interested in a stability result and the arguments developed in the proof of Theorem 3.1 allow us to obtain that up to a subsequence still indexed by ε

$$u_\varepsilon \to u \text{ a.e in } Q_T, \tag{3.50}$$

up to a subsequence still indexed by ε, where u , is a renormalized solution to (2.1) and $\nabla T_k(u_\varepsilon) \to \nabla T_k(u)$ strongly in $L^p((0,T);W_0^{1,p}(\Omega))$, for every $k > 0$.

The crucial point is to obtain the a priori estimates

$$\left\| |u_\varepsilon|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q_T)} \le C_{14}, \tag{3.51}$$

$$\left\| |\nabla u_\varepsilon|^{\delta} \right\|_{L^{\frac{N+2}{N+1},\infty}(Q_T)} \le C_{14}, \tag{3.52}$$

$$\| u_\varepsilon \|_{L^\infty((0,T);L^1(\Omega))} \le C_{14} \tag{3.53}$$

where C_{14} is a constant which depends only on N, p, α, $|Q_T|$, $\|c\|$, $\|b\|$, $\|u_0\|$, $\|f\|$ and $\|g\|$. Following Step 2 of the proof of Theorem 3.1 it remains to obtain inequality (3.9). Even if $T_k(u_\varepsilon)$ is not an admissible test function in the renormalized formulation it is well known that it can be achieved through the following process. Using the admissible test function $S'_n(u_\varepsilon)T_k(S_n(u_\varepsilon))$ in (3.46) where S_n is a sequence of increasing $C^\infty(\mathbb{R})-$function such that

$$S_n(r) = r \quad \text{for } |r| \le n,$$

$$\operatorname{supp} S'_n \subset [-2n, 2n],$$

$$\| S''_n \|_{L^\infty(\mathbb{R})} \le \frac{3}{n}$$

and integrating on $(0,t)$ for almost every $t \in (0,T)$ and we can pass to the limit as n goes to infinity in the resulting equality using the energy condition (2.15) and Remark 2. It follows that for any $k > 0$ and for almost any t in $(0,T)$ we have

$$\begin{aligned}
&\int_\Omega \Psi_k(u_\varepsilon(t))dx + \int_0^t \int_\Omega a(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon)dxdt \\
&\quad + \int_0^t \int_\Omega K_\varepsilon(x,t,u_\varepsilon)\nabla T_k(u_\varepsilon)dxdt + \int_0^t \int_\Omega H_\varepsilon(x,t,\nabla u_\varepsilon)T_k(u_\varepsilon)dxdt \\
&\qquad = \int_\Omega \Psi_k((u_0)_\varepsilon(t))dx + \int_0^t \int_\Omega f_\varepsilon T_k(u_\varepsilon)dxdt + \int_0^t \int_\Omega g_\varepsilon \nabla T_k(u_\varepsilon)dxdt.
\end{aligned}$$

Then the inequality (3.9) holds and we can obtain (3.51)–(3.53).

For the same reasons following Step 3 there exists a subsequence u_ε and a function $u \in L^\infty((0,T);L^1(\Omega))$ such that

$$\nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u) \text{ weakly in } L^p((0,T);W_0^{1,p}(\Omega)),$$

$$u_\varepsilon \to u \text{ a.e. in } Q_T,$$

$$H_\varepsilon(x,t,\nabla u_\varepsilon) \rightharpoonup \Lambda \text{ weakly in } L^1(Q_T),$$

$$a(x,t,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) \rightharpoonup \sigma_k \text{ weakly in } (L^{p'}(Q_T))^N$$

and

$$\lim_{n\to+\infty} \limsup_{\varepsilon\to 0} \frac{1}{n} \int_{\{|u_\varepsilon|<n\}} a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon dxdt = 0.$$

Arguing as in [4] and using the strict monotone character of the operator a allow to prove that u is a renormalized solution to problem (2.1) and that $T_k(u_\varepsilon)$ strongly converges to $T_k(u)$ in $L^p((0,T);W_0^{1,p}(\Omega))$ as ε goes to zero.

Appendix A. Proof of Lemma A.1

Lemma A.1. *Assume that $Q_T = \Omega \times (0,T)$ with Ω open subset of $\mathbb{R}^N$ of finite measure and $p > 1$. Let be u a measurable function satisfying*
$T_k(u) \in L^\infty\left(0,T,L^2(\Omega)\right) \cap L^p\left(0,T,W_0^{1,p}(\Omega)\right)$ *for every $k > 0$ and such that*

$$\sup_{t\in(0,T)} \int_\Omega |T_k(u(t))|^2 + \int_0^T \int_\Omega |\nabla T_k(u)|^p \leq kM + L \tag{A.1}$$

where M and L are constants. Then $u^{\frac{N(p-1)+p}{N+p}} \in L^{\frac{N+p}{N},\infty}(Q_T)$ and $|\nabla u|^{\frac{N(p-1)+p}{N+2}} \in L^{\frac{N+2}{N+1},\infty}(Q_T)$ and

$$\left\| |u|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q_T)} \leq C(N,p)\left[M + |Q_T|^{\frac{Np}{N+2}} L^{\frac{N(p-1)+p}{(N+2)p}}\right] \tag{A.2}$$

$$\left\| |\nabla u|^{\frac{N(p-1)+p}{N+2}} \right\|_{L^{\frac{N+2}{N+1},\infty}(Q_T)} \leq C(N,p)\left[M + |Q_T|^{\frac{N}{(N+2)p}} L^{\frac{N(p-1)+p}{(N+2)p}}\right] \tag{A.3}$$

where $c(N,p)$ is a constant depending only on p and N.

Proof of Lemma A.1. We first prove estimate (A.2). Using Gagliardo-Nirenberg inequality and (A.1) we have

$$\begin{aligned}\int_0^T \int_\Omega |T_k(u)|^{p\frac{N+2}{N}}\,dxdt &\leq C_1 \left(\sup_{t\in(0,T)} \int_\Omega |T_k(u(t))|^2\,dxdt\right)^{\frac{p}{N}} \int_0^T \int_\Omega |\nabla T_k(u)|^p\,dxdt \\ &\leq C_1\,(kM+L)^{\frac{p}{N}+1}\end{aligned}$$

It follows that for every $k > 0$

$$\begin{aligned}C_1\,(kM+L)^{\frac{p}{N}+1} &\geq \int_0^T \int_\Omega |T_k(u)|^{p\frac{N+2}{N}}\,dxdt \\ &\geq k^{\frac{p(N+2)}{N}}\ \mathrm{meas}\{x\in\Omega : |u| > k\}\end{aligned} \tag{A.4}$$

or equivalently (taking $k = h^{\frac{N+p}{N(p-1)+p}}$) , for every $h > 0$

$$h^{\frac{p(N+2)}{N}\frac{N+p}{N(p-1)+p}}\ \mathrm{meas}\{(x,t)\in Q_T : |u|^{\frac{N(p-1)+p}{N+p}} > h\} \leq C_1(Mh^{\frac{N+p}{N(p-1)+p}} + L)^{\frac{p}{N}+1}.$$

We deduce that

$$\mathrm{meas}\{(x,t)\in Q_T : |u|^{\frac{N(p-1)+p}{N+p}} > h\} \leq C_1(Mh^{-1} + Lh^{-\frac{p(N+2)}{N(p-1)+p}})^{\frac{p}{N}+1},$$

and we get

$$h\left(\mathrm{meas}\{(x,t)\in Q_T : |u|^{\frac{N(p-1)+p}{N+p}} > h\}\right)^{\frac{N}{N+p}} \leq C_1^{\frac{N}{N+p}}(M + Lh^{-\frac{N+p}{N(p-1)+p}}),$$

for every $h > 0$.

Therefore we have

$$\begin{aligned}\left\| u^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q_T)} &= \sup_{h>0} h\left(\text{meas}\{(x,t)\in Q_T : |u|^{\frac{N(p-1)+p}{N+p}} > h\}\right)^{\frac{N}{N+p}} \\ &\le \sup_{0<h<h_0} h\left(\text{meas}\{(x,t)\in Q_T : |u|^{\frac{N(p-1)+p}{N+p}} > h\}\right)^{\frac{N}{N+p}} \\ &\quad + \sup_{h>h_0} h\left(\text{meas}\{(x,t)\in Q_T : |u|^{\frac{N(p-1)+p}{N+p}} > h\}\right)^{\frac{N}{N+p}} \\ &\le h_0 |Q_T|^{\frac{N}{N+p}} + C_1^{\frac{N}{N+p}}(M + Lh^{-\frac{N+p}{N(p-1)+p}})\end{aligned}$$

which, taking $h_0 = \dfrac{L^{\frac{N(p-1)+p}{(N+2)p}}}{|Q_T|^{\frac{N}{N+p}\frac{N(p-1)+p}{(N+2)p}}}$, proves (A.2).

The proof of (A.3) is divided into 4 steps.

Step 1. From (A.1) we deduce that for every $\lambda > 0$ and every $k > 0$

$$\begin{aligned}\lambda^p\, &\text{meas}\{(x,t)\in Q_T : |\nabla u| > \lambda \text{ and } |u| < k\} \\ &\le \int\int_{\{|u|<k\}} |\nabla u|^p = \int_0^T \int_\Omega |\nabla T_k(u)|^p\, dxdt \\ &\le Mk + L\end{aligned}$$

i.e. for every $\mu > 0$ and every $k > 0$ (taking $\lambda = \mu^{\frac{N+2}{N(p-1)+p}}$)

$$\begin{aligned}\mu^{\frac{p(N+2)}{N(p-1)+p}}\, &\text{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu \text{ and } |u| < k\} \\ &\le Mk + L.\end{aligned} \tag{A.5}$$

From (A.4) and (A.5) we obtain that, for every $\lambda > 0$ and every $k > 0$

$$\begin{aligned}\text{meas}\{(x,t)\in Q_T : &|\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\} \\ &\le \text{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu \text{ and } |u| < k\} \\ &\quad + \text{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu \text{ and } |u| > k\} \\ &\le \frac{Mk+L}{\mu^{\frac{p(N+2)}{N(p-1)+p}}} + C_1 (kM+L)^{\frac{p}{N}+1} k^{-\frac{p(N+2)}{N}}\end{aligned}$$

Step 2. We now write

$$k = a + b \qquad \text{with} \qquad a > 0,\ b > 0.$$

From the inequality $(x+y)^{\frac{p^*}{p}} \le 2^{\frac{p^*}{p}}(x^{\frac{p^*}{p}} + y^{\frac{p^*}{p}})$, we get

$$\begin{aligned}
&\operatorname{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\\
&\le \frac{Ma}{\mu^{\frac{p(N+2)}{N(p-1)+p}}} + \frac{Mb}{\mu^{\frac{p(N+2)}{N(p-1)+p}}} + \frac{L}{\mu^{\frac{p(N+2)}{N(p-1)+p}}} +\\
&+ C_1 2^{1+\frac{p}{N}}(a+b)^{\frac{p}{N}+1-\frac{p(N+2)}{N}} M^{\frac{p}{N}+1}\\
&+ C_1 2^{1+\frac{p}{N}}(a+b)^{-\frac{p(N+2)}{N}} L^{\frac{p}{N}+1}
\end{aligned}$$

for every $\mu > 0$, $a > 0$ and $b > 0$. Since $(a+b)^{-p^*} \le b^{-p^*}$ and since $(a+b)^{\frac{p^*}{p}-p^*} \le a^{\frac{p^*}{p}-p^*}$ (indeed $\frac{p^*}{p} - p^* = -\frac{p^*}{p'} < 0$), we obtain

$$\begin{aligned}
&\operatorname{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\\
&\le C(N,p)\left\{\left[\frac{Ma}{\mu^{\frac{p(N+2)}{N(p-1)+p}}} + a^{\frac{p}{N}+1-\frac{p(N+2)}{N}} M^{\frac{p}{N}+1}\right]\right.\\
&\left. + \left[\frac{Mb}{\mu^{\frac{p(N+2)}{N(p-1)+p}}} + b^{-\frac{p(N+2)}{N}} L^{\frac{p}{N}+1}\right] + \frac{L}{\mu^{\frac{p(N+2)}{N(p-1)+p}}}\right\}
\end{aligned} \tag{A.6}$$

for some constant $C(N,p)$.

Step 3. For the rest of the present proof, we will denote by $C(N,p)$ a constant which only depends on N and p, but can vary from line to line.

Choosing

$$a = M^{\frac{1}{N+1}} \mu^{\frac{N+2}{(N+1)}\frac{N}{N(p-1)+p}}$$

$$b = \frac{L^{\frac{N+p}{N+p(N+2)}} \mu^{\frac{N}{N+pN+2p}\frac{p(N+2)}{n(p-1)+p}}}{M^{\frac{N}{N+pN+2p}}}$$

(those are the values which minimize with respect to a and b the right-hand side of (A.6)), inequality (A.6) yields

$$\begin{aligned}
&\operatorname{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\\
&\le C(N,p)\left[\frac{M^{\frac{N+2}{N+1}}}{\mu^{\frac{N+2}{N+1}}} + \frac{M^{\frac{p(N+2)}{N+pN+2p}} L^{\frac{N+p}{N+pN+2p}}}{\mu^{\frac{p(N+2)}{(n(p-1)+p)}\left(1-\frac{N}{(N+pN+2p)}\right)}} + \frac{L}{\mu^{\frac{p(N+2)}{N(p-1)+p}}}\right]
\end{aligned}$$

and then

$$\begin{aligned}
&\mu\Big(\operatorname{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\Big)^{\frac{N+1}{N+2}}\\
&\le C(N,p)\left[M + \frac{M^{\frac{(N+1)p}{N+pN+2p}} L^{\frac{N+p}{N+pN+2p}\frac{N+1}{N+2}}}{\mu^{\frac{N(p+N)}{(N+pN+2p)(n(p-1)+p)}}} + \frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{N(p-1)+p}}}\right]
\end{aligned}$$

Let be

$$\frac{1}{q} = \frac{p(N+1)}{N+pN+2p}, \frac{1}{q'} = \frac{N+p}{N+pN+2p}$$

$$\mu\Big(\mathrm{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\Big)^{\frac{N+1}{N+2}}$$
$$\leq C(N,p)\left[M + M^{\frac{1}{q}}\left(\frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{(n(p-1)+p)}}}\right)^{\frac{1}{q'}} + \frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{N(p-1)+p}}}\right]$$

Therefore Young inequality yields

$$\text{(A.7)}\qquad \mu\Big(\mathrm{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\Big)^{\frac{N+1}{N+2}}$$
$$\leq C(N,p)\Big[M + \frac{p(N+1)}{N+pN+2p}M + \frac{N+p}{N+pN+2p}\frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{(n(p-1)+p)}}}$$
$$+\frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{N(p-1)+p}}}\Big]$$
$$\leq C(N,p)\left[M + \frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{(n(p-1)+p)}}}\right]$$

for every $\mu > 0$.

Step 4. From (A.7), we deduce that

$$\text{(A.8)}\qquad \||\nabla u|^{\frac{N(p-1)+p}{N+2}}\|_{L^{\frac{N+2}{N+1},\infty}(Q_T)} = \sup_{\mu>0}\mu\Big(\mathrm{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\Big)^{\frac{N+1}{N+2}}$$
$$\leq \sup_{0<\mu<\mu_0}\mu\Big(\mathrm{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\Big)^{\frac{N+1}{N+2}}$$
$$+\sup_{\mu>\mu_0}\mu\Big(\mathrm{meas}\{(x,t)\in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\}\Big)^{\frac{N+1}{N+2}}$$
$$\leq \mu_0|Q_T|^{\frac{N+1}{N+2}} + C(N,p)\left[M + \frac{L^{\frac{N+1}{N+2}}}{\mu_0^{\frac{N}{(n(p-1)+p)}}}\right]$$
$$\leq C(N,p)\left[\mu_0|Q_T|^{\frac{N+1}{N+2}} + M + \frac{L^{\frac{N+1}{N+2}}}{\mu_0^{\frac{N}{(N(p-1)+p)}}}\right]$$

Choosing

$$\mu_0 = \left(\frac{L}{|Q_T|}\right)^{\frac{N(p-1)+p}{N+2}}$$

(this is the value which minimizes the right-hand side of (A.8) with respect to μ_0) we obtain

$$\sup_{\mu>0} \mu \left(\text{meas}\{(x,t) \in Q_T : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > \mu\} \right)^{\frac{N+1}{N+2}} \le C(N,p) \left[M + |Q_T|^{\frac{N}{(N+2)p}} L^{\frac{N(p-1)+p}{(N+2)p}} \right]$$

which is the desired result. □

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