INFINITE VALUED SOLUTIONS OF NON UNIFORMLY ELLIPTIC PROBLEMS

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ABSTRACT. We consider a quasilinear equation (see (1.1)) with L^1 data and with a diffusion matrix $\mathbf{A}(x,u)$ which is not uniformly coercive with respect to u (see Assumptions (H3)–(H4)). Under such assumptions it is not realistic, in general, to search a solution which is finite almost everywhere. We introduce two equivalent notions of solutions which take into account the possible values $+\infty$ and $-\infty$ (see Definitions 2.1 and 2.3). Then we prove that there exists at least one such solution. At last we establish an uniqueness result in the class of simultaneous infinite valued solutions.

1. INTRODUCTION

This paper is devoted to study a class of possibly degenerate elliptic problems of the type

$$\begin{cases} -\operatorname{div}(\mathbf{A}(x,u)Du) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 1$), $\mathbf{A}(x,s)$ is a Carathéodory function with symmetric matrix values and f belongs to $L^1(\Omega)$. For almost any x in Ω, the matrix $\mathbf{A}(x,s)$ "strongly" degenerates when $|s| \to +\infty$ (with a kind of uniform dependence with respect to x; see assumption (H3) and (H4)) so that we cannot avoid solutions of (1.1) (at least obtained through approximation processes) to reach the values $+\infty$ and $-\infty$. A model case of (1.1) is to consider $\mathbf{A}(x,u) = \frac{\lambda(x)}{(1+|u|)^m}$ where $\lambda(x) \in L^\infty(\Omega)$ with $0 < \lambda_0 \leq \lambda(x)$ almost everywhere in Ω and $m > 1$. Such cases have been examined in [1], [2] and [7] for $f \in L^p(\Omega)$ (with $p > N/2$) and $\|f\|_{L^p(\Omega)}$ small enough (with also some extensions to nonlinear operators $a(x,u,Du)$). In these papers the assumptions on the data lead to finite (almost everywhere in Ω) solutions. In [1] the reader could find an example where the explicit behavior of the bounded solution obtain for λf for λ small (in a specific geometry of Ω) is investigated when λ increases. The authors show that there exists a critical value λ^* such that the solution u_λ reaches the value $+\infty$ for $\lambda > \lambda^*$. Let us emphasis that in the present paper we propose a formulation which takes into account the possible values $+\infty$ or $-\infty$ for the solutions.

Key words and phrases. existence, non uniformly elliptic problem, integrable data, infinite valued solution.

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As an example, let us consider the very simple case where $\mathbf{A}(x,s) = a(s)I$ where $a(s)$ is a positive continuous function defined on $\mathbb{R}$ with values in $\mathbb{R}$ and is such that $\int_{-\infty}^{+\infty} a(s)ds < +\infty$ (see assumption (H3) and (H4)). Let $\widetilde{a}(t) = \int_0^t a(s)ds$ which is then a $\mathscr{C}^1$ bounded function on $\mathbb{R}$. If one formally rewrites (1.1) in this case as $-\Delta\widetilde{a}(u) = f$ in Ω and $u = 0$ on $\partial\Omega$, there is no hope, even for $f \in L^2(\Omega)$, to find a solution u since indeed $\widetilde{a}(u)$ should be equal to the unique solution $v \in H_0^1(\Omega)$ of $-\Delta v = f$ and v is not in the range of $\widetilde{a}$ (in general). Now if one consider the approximate equation $-\operatorname{div}\big(a(u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big) = f$ for $f \in L^1(\Omega)$, we have $a(u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon = Dv$ where $Dv \in L^q(\Omega)$ for any $1 \le q < N(N-1)$. Moreover it is well known that any truncation $T_k(v)$ of v belongs to $H_0^1(\Omega)$. Since $\widetilde{a}^\varepsilon(u^\varepsilon) = v$, where $\widetilde{a}^\varepsilon(t) = \int_0^t a(s)ds + \varepsilon t$, one has $u^\varepsilon \to u$ almost everywhere in Ω where $u(x) = +\infty$ if $v(x) \ge \sup_{t\ge 0}\widetilde{a}(t)$, $u(x) = -\infty$ if $v(x) \le \inf_{t\le 0}\widetilde{a}(t)$ and $u(x) = (\widetilde{a})^{-1}(v(x))$ if $v(x)$ belongs to the range of $\widetilde{a}$. Then we have $\mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}Dv \in (L^2(\Omega))^N$. Now for any $k \ge 0$, passing to the limit as ε tends to 0 in the relation $a(u^\varepsilon)DT_k(u^\varepsilon) = \frac{a(u^\varepsilon)}{a(u^\varepsilon)+\varepsilon}T_k'(u^\varepsilon)Dv$ leads to $a(u)DT_k(u) = T_k'(u)Dv$ almost everywhere in $\Omega \setminus \{x \in \Omega;\, |u(x)| = k\}$. Letting k tend to infinity gives that $\mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}a(u)Du = \mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}Dv$ almost everywhere in Ω, which shows that $\mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}a(u)Du \in (L^2(\Omega))^N$ (because $\mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}Dv = \mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}DT_M(v) \in (L^2(\Omega))^N$ for M large enough). This property remains true in the general case (1.1) under assumption (H2)–(H5) (see 2.7)). The equation $-\Delta v = f$ is then written in terms of u as

$$-\operatorname{div}\big[\mathbb{1}_{\{x\in\Omega;\,|u(x)|<+\infty\}}a(u)Du\big] - \operatorname{div}\big[\mathbb{1}_{\{x\in\Omega;\,|u(x)|=+\infty\}}Dv\big] = f \quad \text{in } \Omega \tag{1.2}$$

but in general the term $\mathbb{1}_{\{x\in\Omega;\,|u(x)|=+\infty\}}Dv$ can not be explicitly expressed in terms of u.

In this paper we propose two definitions of a solution for Problem (1.1). The first notion of solutions (Subsection 2.2.1) uses a renormalized formulation of the equation, which is formally obtained by using test functions of the form $h(u)\varphi$ in (1.1) where h has a compact support. As usual two conditions on the asymptotic behavior of the energy when $|u|$ is large with respect to the subsets $\{x \in \Omega;\, u(x) = +\infty\}$ and $\{x \in \Omega;\, u(x) = -\infty\}$ are imposed (see conditions (2.2)–(2.3)). As far as renormalized solutions are concerned we refer to [10], [9], [11], [12] and [13]. The second notion of solution (Section 2.2.2) follows from a generalization of (1.2) through an evaluation of the quantity $\operatorname{div}\big[\mathbb{1}_{\{x\in\Omega;\,|u(x)|=+\infty\}}Dv\big]$ in terms of two measures which are, loosely speaking, supported by the two subsets $\{x \in \Omega;\, u(x) = +\infty\}$ and $\{x \in \Omega;\, u(x) = -\infty\}$ of Ω (in this sense this formulation is formally similar to the one used in [9]).

We first show that the two notions of solutions are equivalent (Section 3). Then, in Section 4, we prove the existence of at least one solution under the assumptions (H1)–(H5). At last, in Section 5, we establish a partial uniqueness result : all the solutions which are infinite almost everywhere on the same subset of Ω are equal.

2. Assumptions on the data and Definitions of a solution

2.1. Assumptions.

We assume that the data of the problem (1.1) satisfy the following assumptions:

- (H1) $\mathbf{A}(x,s) : \Omega \times \mathbb{R} \mapsto \mathbb{R}^{N\times N}$ is a field of symmetric matrices with coefficients $(\mathbf{A}_{ij})_{1\le i,j\le N}$ such that $\mathbf{A}_{i,j}(x,s) \in L^\infty(\Omega)$, $\forall s \in \mathbb{R}$;
- (H2) $\mathbf{A}_{ij}$ is a Carathéodory function defined on $\Omega \times \mathbb{R}$ and $\forall k > 0$, $\mathbf{A}_{ij} \in L^\infty(\Omega \times]-k,k[)$;
- (H3) there exists a positive function $\beta \in \mathscr{C}^0(\mathbb{R})$ such that

$$\beta(s)|\xi|^2 \le \mathbf{A}(x,s)\xi\cdot\xi, \quad \forall s \in \mathbb{R};\ \forall \xi \in \mathbb{R}^N \text{ almost everywhere in } \Omega;$$

(H4) there exists a positive function $\gamma \in \mathscr{C}^0(\mathbb{R})$ satisfying

$$\int_{-\infty}^{+\infty} \gamma(s)ds < +\infty$$

and such that

$$\mathbf{A}(x,s)\xi\cdot\xi \le \gamma(s)|\xi|^2, \quad \forall s \in \mathbb{R};\ \forall \xi \in \mathbb{R}^N \text{ almost everywhere in } \Omega;$$

(H5) $f \in L^1(\Omega)$.

Remark that assumptions (H3)–(H4) imply that the operator $-\operatorname{div}\left[\mathbf{A}(x,u)Du\right]$ is strongly degenerated at both $+\infty$ and $-\infty$. This means that, at least for solutions obtained through approximation, such solutions may reach the values $+\infty$ and $-\infty$. Indeed we can also deal with operators which degenerate only at $+\infty$ or at $-\infty$ (see Remark 2.2)

2.2. **Definitions of a solution.** In this section we give two notions of solution for Problem (1.1) and a few comments on these definitions. Let us first set a few notations. For any measurable subset E of Ω, we denote by meas(E) the Lebesgue measure of E. For any measurable function v defined on Ω with value in $\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ and for any $s \in \mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$, $\mathbb{1}_{\{v<s\}}$ (respectively $\mathbb{1}_{\{v=s\}}$, $\mathbb{1}_{\{v>s\}}$) is the characteristic function of the set $\{x\in\Omega;\ v(x)<s\}$ (respectively $\{x\in\Omega;\ v(x)=s\}$, $\{x\in\Omega;\ v(x)>s\}$). For any real number $k\ge 0$, $T_k(s)$ is the truncation at height $\pm k$: $T_k(s) = \max(-k,\min(s,k))$. For any $n\in\mathbb{N}$ we denote by h_n the Lipschitz continuous function defined on $\mathbb{R}$ by

$$h_n(s) = 1 - |T_{n+1}(s) - T_n(s)|, \quad \forall s \in \mathbb{R}.$$

2.2.1. *Renormalized solution of* (1.1).

Definition 2.1. A measurable function u defined on Ω with value in $\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ is a renormalized solution of (1.1) if

$$\forall k \ge 0, \quad T_k(u) \in H_0^1(\Omega); \tag{2.1}$$

for any function $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $D\varphi = 0$ a.e. on $\{x\in\Omega;\ u(x)=+\infty\}$

$$\lim_{n\to+\infty}\int_\Omega \mathbb{1}_{\{n<u<n+1\}}\mathbf{A}(x,u)Du\cdot Du\varphi\,\mathrm{d}x = \int_{\{u=+\infty\}} f\varphi\,\mathrm{d}x; \tag{2.2}$$

for any function $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $D\varphi = 0$ a.e. on $\{x\in\Omega;\ u(x)=-\infty\}$

$$\lim_{n\to+\infty}\int_\Omega \mathbb{1}_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)Du\cdot Du\varphi\,\mathrm{d}x = -\int_{\{u=-\infty\}} f\varphi\,\mathrm{d}x; \tag{2.3}$$

for any function $h\in W^{1,\infty}(\mathbb{R})$ such that supp(h) is compact, u satisfies the equation

$$-\operatorname{div}\left[h(u)\mathbf{A}(x,u)Du\right] + h'(u)\mathbf{A}(x,u)Du\cdot Du = h(u)f \quad \text{in } \mathscr{D}'(\Omega). \tag{2.4}$$

Comments on Definition 2.1. In (2.1), the Lipschitz continuous function T_k is extended to $\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ by setting $T_k(+\infty)=k$ and $T_k(-\infty)=-k$. Then $T_k(u)$ makes sense for $u:\Omega\mapsto\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ and $T_k(u)$ belongs to $L^\infty(\Omega)$ for a measurable function u defined on Ω.

As usual when dealing with renormalized solution, condition (2.1) permits to give a sense in all the terms entering in (2.2), (2.3) and (2.4). For all real numbers a, b with $a<b$, $\mathbb{1}_{\{a<u<b\}}Du$ is identified with $\mathbb{1}_{\{a<u<b\}}DT_{\max(|a|,|b|)}(u)$ which belongs to $(L^2(\Omega))^N$ in view of (2.1). This is the definition of Du on the subset $\{x\in\Omega;\ |u(x)|<+\infty\}$ which is introduced in [4]. Then $\mathbb{1}_{\{n<u<n+1\}}\mathbf{A}(x,u)Du\cdot Du$ (resp. $\mathbb{1}_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)Du\cdot Du$) is identified with $\mathbb{1}_{\{n<u<n+1\}}\mathbf{A}(x,u)DT_{n+1}(u)\cdot DT_{n+1}(u)$

(resp. $\mathbb{1}_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)DT_{n+1}(u)\cdot DT_{n+1}(u)$) which belongs to $L^1(\Omega)$ because of (H2) and (2.1). It follows that conditions (2.2) and (2.3) make sense. As far as (2.4) is concerned, let us denote by $k>0$ a real number such that $\operatorname{supp}(h)\subset[-k,k]$ so that $h(u)\mathbf{A}(x,u)Du$ is identified with $h(u)\mathbb{1}_{\{|u|<k\}}\mathbf{A}(x,T_k(u))DT_k(u)$ which belongs to $(L^2(\Omega))^N$. Then the term $h'(u)\mathbf{A}(x,u)Du\cdot Du$ is identified with $\mathbb{1}_{\{|u|<k\}}\mathbf{A}(x,T_k(u))Dh(T_k(u))\cdot DT_k(u)$ which belongs to $L^1(\Omega)$ because of (H2), (2.1) and $h\in W^{1,\infty}(\mathbb{R})$. As a consequence the equation in (2.4) makes sense in $H^{-1}(\Omega)+L^1(\Omega)$ (indeed $fh(u)\in L^1(\Omega)$) and then in the sense of distribution. The boundary condition $u=0$ on $\partial\Omega$ is written in the weak sense : $T_k(u)=0$ on $\partial\Omega$ for any $k\geq 0$. Any test function $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ may be plugged in (2.4). Then if H is a function in $W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp}H'$ is compact, one can use $H(u)$ as such a test function because $H(u)\in L^\infty(\Omega)$ and $DH(u)$ is identified with $DH(T_k(u))=H'(T_k(u))DT_k(u)\in L^2(\Omega)$ (for any k such that $\operatorname{supp}(H')\in[-k,k]$).

Equation (2.4) may be formally obtained through pointwise multiplication of the equation of (1.1) by $h(u)$. Indeed it results a lack of information on the subset $\{x\in\Omega;|u(x)|=+\infty\}$ since $\operatorname{supp}(h)$ is compact. This is balanced by the two conditions (2.2) and (2.3). The same type of conditions is prescribed when dealing with measure data (for non degenerate elliptic operators) in [9] but one must remark that in our setting the Lebesgue measure of the set $\{x\in\Omega;|u(x)|=+\infty\}$ may not vanish. Moreover taking the admissible function $\varphi=T_1(u)^+$ in (2.3) gives that

$$\lim_{n\to+\infty}\int_\Omega \mathbb{1}_{\{n<u<n+1\}}\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x=\int_{\{u=+\infty\}} fT_1(u)^+\,\mathrm{d}x=\int_{\{u=+\infty\}} f\,\mathrm{d}x. \tag{2.5}$$

Similarly we also have

$$\lim_{n\to+\infty}\int_\Omega \mathbb{1}_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x=-\int_{\{u=-\infty\}} f\,\mathrm{d}x. \tag{2.6}$$

Remark 2.2. If assumptions (H3) and (H4) are replaced by $\int_0^{+\infty}\beta(s)\,\mathrm{d}s=\int_0^{+\infty}\gamma(s)\,\mathrm{d}s=+\infty$, $\int_{-\infty}^0\beta(s)\,\mathrm{d}s<+\infty$ and $\int_{-\infty}^0\gamma(s)\,\mathrm{d}s<+\infty$ (respectively $\int_{-\infty}^0\beta(s)\,\mathrm{d}s=\int_{-\infty}^0\gamma(s)\,\mathrm{d}s=+\infty$, $\int_0^{+\infty}\beta(s)\,\mathrm{d}s<+\infty$ and $\int_0^{+\infty}\gamma(s)\,\mathrm{d}s<+\infty$) it is easy to see that one can ask $u(x)<+\infty$ almost everywhere in Ω and take 0 in the right hand side of (2.2) (respectively $-\infty<u(x)$ almost everywhere in Ω; 0 in the right hand side of (2.3)).

2.2.2. *A second definition of solutions of* (1.1). In this subsection we introduce a formulation of (1.1) which takes into account the two subsets $\Omega^{+\infty}=\{x\in\Omega;u(x)=+\infty\}$ and $\Omega^{-\infty}=\{x\in\Omega;u(x)=-\infty\}$ through two bounded measures in the equation of (1.1). Loosely speaking these two measures are supported on $\Omega^{+\infty}$ and $\Omega^{-\infty}$. We denote by $\mathcal{M}(\Omega)$ the set of bounded Radon measures on Ω.

Definition 2.3. A measurable function $u:\Omega\mapsto\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ is a solution of (1.1) (in the sense of Definition 2.3) if u satisfies (2.1) and if

$$\mathbb{1}_{\{|u|<\infty\}}\mathbf{A}(x,u)Du\in(L^2(\Omega))^N \tag{2.7}$$

and if there exist two non-negative bounded measures μ^+ and μ^- on Ω such that

$$\mu^+,\mu^-\in\left(H^{-1}(\Omega)+L^1(\Omega)\right)\cap\mathcal{M}(\Omega), \tag{2.8}$$

$$u=+\infty\quad \mu^+\text{–almost everywhere}, \tag{2.9}$$

$$u=-\infty\quad \mu^-\text{–almost everywhere}, \tag{2.10}$$

$\forall \varphi \in L^\infty(\Omega)\cap H_0^1(\Omega)$ such that $D\varphi = 0$ a.e. on $\{x\in\Omega;\, u(x)=+\infty\}$ we have

$$\int_\Omega \varphi \,\mathrm{d}\mu^+ = \int_{\{u=+\infty\}} f\varphi\,\mathrm{d}x, \tag{2.11}$$

$\forall \varphi \in L^\infty(\Omega)\cap H_0^1(\Omega)$ such that $D\varphi = 0$ a.e. on $\{x\in\Omega;\, u(x)=-\infty\}$ we have

$$\int_\Omega \varphi \,\mathrm{d}\mu^- = -\int_{\{u=-\infty\}} f\varphi\,\mathrm{d}x, \tag{2.12}$$

and u, μ^+ and μ^- satisfy the equation

$$-\operatorname{div}\left[\mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\right] - \mu^+ + \mu^- = f\,\mathbb{1}_{\{|u|<+\infty\}} \text{ in } \mathscr{D}'(\Omega). \tag{2.13}$$

Comments on Definition 2.3. As shown in [4], condition (2.1) permits to give a sense to the field $\mathbb{1}_{\{|u|<+\infty\}}Du$ as a measurable function on Ω in such a way that for any $k\ge 0$

$$\mathbb{1}_{\{|u|\le k\}}\mathbb{1}_{\{|u|<+\infty\}}Du = DT_k(u). \tag{2.14}$$

Then condition (2.7) makes sense.

Conditions (2.9) and (2.10) say that the two subsets $\Omega^{+\infty}$ and $\Omega^{-\infty}$ are μ–measurable and that $\mu^+(\Omega\setminus\Omega^{+\infty}) = \mu^-(\Omega\setminus\Omega^{-\infty}) = 0$. Let us note that $\Omega^{+\infty}$ and $\Omega^{-\infty}$ are μ–measurable because of (2.1) and (2.8). Actually by a result of [9], the function $T_k(u)$ is μ–measurable for any k while $\Omega^{+\infty} = \cap_{n\in\mathbb{N}}\{x\in\Omega;\, T_n(u)=n\}$ and $\Omega^{-\infty} = \cap_{n\in\mathbb{N}}\{x\in\Omega;\, T_n(u)=-n\}$. As a consequence (2.9) and (2.10) mean that μ^+ and μ^- are respectively concentrated on the subsets $\Omega^{+\infty}$ and $\Omega^{-\infty}$. Moreover for any function $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$ we have in view of Proposition 2.7 of [9],

$$\begin{cases} \displaystyle\int_\Omega v\,\mathrm{d}\mu^+ = <\mu^+, v>_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} \\ \text{and} \\ \displaystyle\int_\Omega v\,\mathrm{d}\mu^- = <\mu^-, v>_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} \end{cases} \tag{2.15}$$

Then (2.9) and (2.10) may be equivalently rewritten as

$$\int_\Omega T_k(u)^+\,\mathrm{d}\mu^+ = <\mu^+, T_k(u)^+>_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} = k\int_\Omega \mathrm{d}\mu^+ \quad \forall k\ge 0 \tag{2.16}$$

and

$$\int_\Omega T_k(u)^-\,\mathrm{d}\mu^- = <\mu^-, T_k(u)^->_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} = k\int_\Omega \mathrm{d}\mu^- \quad \forall k\ge 0 \tag{2.17}$$

or

$$T_k(u)^+ = k,\ \mu^+\text{–almost everywhere},\ \forall k\ge 0, \tag{2.18}$$

$$T_k(u)^- = k,\ \mu^-\text{–almost everywhere},\ \forall k\ge 0. \tag{2.19}$$

Conditions (2.11) and (2.12) play the roles of conditions (2.2)–(2.3) in Definition 2.1 since we will prove in Section 3 that μ^+ and μ^- are the weak–$*$ limit in $\mathscr{M}(\Omega)$ of the respective sequences $\mathbb{1}_{\{n<u<n+1\}}\mathbf{A}(x,u)Du\cdot Du$ and $\mathbb{1}_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)Du\cdot Du$ as n tends to $+\infty$.

As far as Equation (2.13) is concerned, let us note that this equation takes place in $H^{-1}(\Omega) + L^1(\Omega)$ (hence in $\mathscr{D}'(\Omega)$). As a consequence any test function in $H_0^1(\Omega)\cap L^\infty(\Omega)$ can be used in (2.13).

Equation of type (2.13) with multiplier mesures has been also introduced in [6] in the case where the matrix $\mathbf{A}(x,s)$ blows up for a finite value of s.

3. Equivalence of the definitions

We prove the following lemma

Lemma 3.1. *Definition 2.1 is equivalent to Definition 2.3.*

Remark 3.2. According to Lemma 3.1, a solution of (1.1) in the sense of Definition 2.1 or Definition 2.3 will be just referred to as a renormalized solution of the problem.

Proof of Lemma 3.1.
Step 1. Let u be solution of (1.1) in the sense of Definition 2.1 and let us prove that u satisfies Definition 2.3.

Let us choose $h = h_n$ in Equation (2.4), we obtain

$$-\operatorname{div}\big[h_n(u)\mathbf{A}(x,u)Du\big] - 1\!\!1_{\{n<u<n+1\}}\mathbf{A}(x,u)Du\cdot Du + 1\!\!1_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)Du\cdot Du = h_n(u)f \quad \text{in } \Omega. \tag{3.1}$$

We first prove (2.7). The function $v_k = \int_0^{T_k(u)}\gamma(s)\,\mathrm{d}s$ is an admissible test function in (3.1) because $v_k \in L^\infty(\Omega)\cap H_0^1(\Omega)$. Moreover we have $\|v_k\|_{L^\infty(\Omega)} \le \max\big(\int_0^{+\infty}\gamma(s)\,\mathrm{d}s, \int_{-\infty}^0\gamma(s)\,\mathrm{d}s\big)$ for any $k\ge 0$. We obtain using (2.5) and (2.6)

$$\int_\Omega \gamma\big(T_k(u)\big)h_n(u)\mathbf{A}(x,u)Du\cdot DT_k(u)\,\mathrm{d}x \le C_1\|f\|_{L^1(\Omega)} + C_2$$

where C_1 and C_2 are two nonnegative constants independent of k and n. Since $DT_k(u) = 0$ on the subset $\{x\in\Omega;\,|u(x)>k\}$, this yields for $n>k$

$$\int_\Omega \gamma\big(T_k(u)\big)\mathbf{A}(x,T_k(u))DT_k(u)\cdot DT_k(u)\,\mathrm{d}x \le C_1\|f\|_{L^1(\Omega)} + C_2.$$

The condition (H4) permits us to obtain

$$\int_\Omega |\mathbf{A}(x,T_k(u))DT_k(u)|^2\,\mathrm{d}x \le C_1\|f\|_{L^1(\Omega)} + C_2.$$

Indeed we have $\gamma(T_k(u))\mathbf{A}(x,T_k(u))DT_k(u)\cdot DT_k(u) \ge \mathbf{A}(x,T_k(u))\mathbf{A}^{1/2}(x,T_k(u))DT_k(u)\cdot\mathbf{A}^{1/2}(x,T_k(u))DT_k(u) = |\mathbf{A}(x,T_k(u))DT_k(u)|^2$.

Since $A(x,T_k(u))DT_k(u) \to 1\!\!1_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du$ almost everywhere in $\Omega\setminus\{|u(x)| = +\infty\}$ as k tends to $+\infty$, it follows that

$$\int_\Omega 1\!\!1_{\{|u|<+\infty\}}|\mathbf{A}(x,u)Du|^2\,\mathrm{d}x \le C_1\|f\|_{L^1(\Omega)} + C_2,$$

and (2.7) holds true.

Now we construct the two measures μ^+ and μ^- such that (2.8), (2.9), (2.10), (2.11) and (2.13) hold true.

Due to conditions (2.2)–(2.3) (see also (2.5) and (2.6) in Comments on Definition 2.1), the sequences of nonnegative $L^1(\Omega)$–functions $\mu_n^+ = 1\!\!1_{\{n<u<n+1\}}\mathbf{A}(x,u)Du\cdot Du$ and $\mu_n^- = 1\!\!1_{\{-(n+1)<u<-n\}}\mathbf{A}(x,u)Du\cdot Du$ are bounded in $L^1(\Omega)$ with respect to n. As a consequence there exists two nonnegative bounded measures μ^+ and μ^- such that, for a subsequence still indexed by n,

$$\begin{cases} \mu_n^+ \rightharpoonup \mu^+ \text{ weakly–}* \text{ in } \mathscr{M}(\Omega),\\ \mu_n^- \rightharpoonup \mu^- \text{ weakly–}* \text{ in } \mathscr{M}(\Omega).\end{cases} \tag{3.2}$$

Then we pass to the limit as n tends to $+\infty$ in (3.1) and we obtain

$$-\operatorname{div}\big[1\!\!1_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\big] - \mu^+ + \mu^- = f\,1\!\!1_{\{|u|<+\infty\}} \quad \text{in } \mathscr{D}'(\Omega),$$

because $\mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du$ and f respectively belong to $(L^2(\Omega))^N$ and $L^1(\Omega)$. The function u and the measures μ^+ and μ^- satisfy (2.13). Moreover we deduce from (3.1) that

$$\mu_n = \mu_n^+ - \mu_n^- \to \mu^+ - \mu^- \quad \text{strongly in } H^{-1}(\Omega) + L^1(\Omega). \tag{3.3}$$

Now for any $k \geq 0$, $T_k(u)^+$ belongs to $H_0^1(\Omega)$, hence taking $T_k(u)^+\varphi$, for $\varphi \in \mathscr{C}_0^\infty(\Omega)$, as a test function in (3.1) gives

$$\int_\Omega h_n(u)\mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du \cdot D[T_k(u)^+\varphi]\,\mathrm{d}x - \int_\Omega \mu_n T_k(u)^+\varphi\,\mathrm{d}x = \int_\Omega h_n(u) f T_k(u)^+\varphi\,\mathrm{d}x.$$

Using the definition of μ_n, (2.7) and (3.2), it yields letting n tends to $+\infty$

$$\forall\varphi \in \mathscr{C}_0^\infty(\Omega), \quad k\int_\Omega \varphi\,\mathrm{d}\mu^+ = \int_\Omega \mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du \cdot D[T_k(u)^+\varphi]\,\mathrm{d}x - \int_\Omega f T_k(u)^+\varphi\,\mathrm{d}x.$$

As a consequence we deduce that μ^+ belongs to $H^{-1}(\Omega) + L^1(\Omega)$, that is (2.8) for μ^+.

Indeed using $T_k(u)^-\varphi$ as a test function in (3.1) leads to (2.8) for μ^-.

We now prove (2.9) and (2.10) in their equivalent formulation (2.18) and (2.19). To this end, we deduce from (2.15) and (3.3) that $\forall\varphi \in \mathscr{C}_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega T_k(u)^+\varphi(\mu_n^+ - \mu_n^-)\,\mathrm{d}x &= < \mu_n^+ - \mu_n^-, T_k(u)^+\varphi >_{H^{-1}(\Omega)+L^1(\Omega), H_0^1(\Omega)\cap L^\infty(\Omega)} \\ &\longrightarrow < \mu^+ - \mu^-, T_k(u)^+\varphi >_{H^{-1}(\Omega)+L^1(\Omega), H_0^1(\Omega)\cap L^\infty(\Omega)}, \quad \text{as } n \to +\infty.\end{aligned} \tag{3.4}$$

Since $\int_\Omega T_k(u)^+\varphi(\mu_n^+ - \mu_n^-)\,\mathrm{d}x = k\int_\Omega \varphi\mu_n^+\,\mathrm{d}x$, (3.2) and (3.4) lead to

$$\forall\varphi \in \mathscr{C}_0^\infty(\Omega); \quad k\int_\Omega \varphi\,\mathrm{d}\mu^+ = < \mu^+ - \mu^-, T_k(u)^+\varphi >_{H^{-1}(\Omega)+L^1(\Omega), H_0^1(\Omega)\cap L^\infty(\Omega)}.$$

Using again (2.8) and (2.15) we obtain

$$\forall\varphi \in \mathscr{C}_0^\infty(\Omega); \quad k\int_\Omega \varphi\,\mathrm{d}\mu^+ = \int_\Omega T_k(u)^+\varphi\,\mathrm{d}\mu^+ - \int_\Omega T_k(u)^+\varphi\,\mathrm{d}\mu^-. \tag{3.5}$$

Remark that in the above equality we have use the fact that μ^+ and μ^- both belong to $(H^{-1}(\Omega) + L^1(\Omega)) \cap \mathscr{M}(\Omega)$ so that $T_k(u)^+\varphi \in L^\infty(\Omega;\mathrm{d}\mu^+) \cap L^\infty(\Omega;\mathrm{d}\mu^-)$.

From (3.5) we deduce that

$$\forall\varphi \in \mathscr{C}_0^\infty(\Omega), \varphi \geq 0; \quad \int_\Omega \big(T_k(u)^+ - k\big)\varphi\,\mathrm{d}\mu^+ \geq 0,$$

because $\mathrm{d}\mu^- \geq 0$.

Since $T_k(u)^+ \leq k$ μ–almost everywhere in Ω, this implies that

$$T_k(u)^+ = k \quad \mu^+\text{–almost everywhere in } \Omega, \tag{3.6}$$

which proves (2.18).

Testing (3.3) with $T_k(u)^-\varphi$ for $\varphi \in \mathscr{C}_0^\infty(\Omega)$ and using a similar argument lead to

$$T_k(u)^- = k \quad \mu^-\text{–almost everywhere in } \Omega, \tag{3.7}$$

which proves (2.19).

To show that (2.11) is satisfied, let φ be an element of $H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $D\varphi=0$ almost everywhere on $\{x\in\Omega;\, u(x)=+\infty\}$. Since $T_1(u)^+$ belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$, the function $T_1(u)^+\varphi$ is admissible in (3.4) (with $k=1$). According to the definitions of μ_n^+ and μ_n^- we obtain

$$\int_\Omega \varphi\mu_n^+\,\mathrm{d}x \longrightarrow < \mu^+-\mu^-, T_1(u)^+\varphi >_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} \text{ as } n \text{ tends to } +\infty. \tag{3.8}$$

Now using again (2.15), we have

$$< \mu^+-\mu^-, T_1(u)^+\varphi >_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} = \int_\Omega T_1(u)^+\varphi\,\mathrm{d}\mu^+ - \int_\Omega T_1(u)^+\varphi\,\mathrm{d}\mu^-,$$

which in view of (3.6)–(3.7) implies that

$$< \mu^+-\mu^-, T_1(u)^+\varphi >_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)} = \int_\Omega \varphi\,\mathrm{d}\mu^+. \tag{3.9}$$

From (3.8), (3.9) it follows that

$$\int_\Omega \mu_n^+\varphi\,\mathrm{d}x \longrightarrow \int_\Omega \varphi\,\mathrm{d}\mu^+ \quad \text{as } n \text{ tends to } +\infty. \tag{3.10}$$

The definition of μ_n^+, condition (2.2) of Definition 2.1 and (3.10) give (2.11) for μ^+ of Definition 2.3.

Replacing $T_1(u)^+\varphi$ by $T_k(u)^-\varphi$ (with $D\varphi=0$ on $\{x\in\Omega;\, u(x)=-\infty\}$) in the above arguments indeed leads to (2.12) for μ^-.

As a conclusion u is a solution in the sense of Definition 2.3.

Step 2. Let u be a solution in the sense of Definition 2.3 and let us prove that u satisfies Definition 2.1.

All we have to show is that u satisfies (2.2), (2.3) and (2.4).

To prove that (2.2) holds true, we plug the admissible test function $\big(T_{n+1}(u)-T_n(u)\big)^+\varphi$ with $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ in (2.13) and we obtain for any $n\geq 1$

$$\begin{aligned}
&\int_\Omega \mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\cdot D\big(T_{n+1}(u)-T_n(u)\big)^+\varphi\,\mathrm{d}x\\
&\qquad + \int_\Omega \mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\cdot D\varphi\big(T_{n+1}(u)-T_n(u)\big)^+\,\mathrm{d}x\\
&\qquad - \langle \mu^+, \big(T_{n+1}(u)-T_n(u)\big)^+\varphi\rangle_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)}\\
&\qquad + \langle \mu^-, \big(T_{n+1}(u)-T_n(u)\big)^+\varphi\rangle_{H^{-1}(\Omega)+L^1(\Omega),H_0^1(\Omega)\cap L^\infty(\Omega)}\\
&\qquad\qquad = \int_\Omega f\,\mathbb{1}_{\{|u|<+\infty\}}\big(T_{n+1}(u)-T_n(u)\big)^+\varphi\,\mathrm{d}x.
\end{aligned}$$

Now due to (2.9), (2.10) and (2.15), it follows that

$$\begin{aligned}
&\int_{\{n<u<n+1\}} \mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\cdot Du\varphi\,\mathrm{d}x\\
&\qquad + \int_\Omega \mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\cdot D\varphi\big(T_{n+1}(u)-T_n(u)\big)^+\,\mathrm{d}x - \int_\Omega \varphi\,\mathrm{d}\mu^+\\
&\qquad\qquad = \int_\Omega f\,\mathbb{1}_{\{|u|<+\infty\}}\big(T_{n+1}(u)-T_n(u)\big)^+\varphi\,\mathrm{d}x.
\end{aligned}$$

Since the sequence $\mathbb{1}_{\{|u|<+\infty\}}\big(T_{n+1}(u)-T_n(u)\big)^+$ converges to 0 almost everywhere in Ω as n tends to $+\infty$ and since $\mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du$ belongs to $(L^2(\Omega))^N$, Lebesgue convergence theorem allows to pass to the limit in the above equality which leads to

$$\lim_{n\to\infty}\int_{\{n<u<n+1\}} \mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du\cdot Du\varphi\,\mathrm{d}x=\int_\Omega \varphi\,\mathrm{d}\mu^+. \tag{3.11}$$

If we now assume that $D\varphi=0$ on $\{u=+\infty\}$, comparing (2.11) and (3.11) shows that (2.2) holds true.

Using the test function $\big(T_{n+1}(u)-T_n(u)\big)^-\varphi$ in (2.13), condition (2.12) and proceeding as above leads to (2.3).

To prove (2.4), let h be in $W^{1,\infty}(\mathbb{R})$ with compact support and φ be in $\mathscr{C}_0^\infty(\Omega)$. Plugging the test function $h(u)\varphi$ in (2.13) gives

$$\forall\varphi\in\mathscr{C}_0^\infty(\Omega),\quad \int_\Omega \mathbf{A}(x,u)Du\cdot D[h(u)\varphi]\,\mathrm{d}x=\int_\Omega f\mathbb{1}_{\{|u|<+\infty\}}h(u)\varphi\,\mathrm{d}x$$

because of (2.9), (2.10) and since $\mathrm{supp}(h)$ is compact. This implies that (2.4) holds true. □

4. EXISTENCE OF A SOLUTION

We prove the following theorem.

Theorem 4.1. *Under the assumptions* (H1)–(H5) *there exists at least a renormalized solution of* (1.1).

Proof of Theorem 4.1. The proof is performed in 4 steps. In Step 1, a sequence of renormalized solution (u^ε) of approximate problems is introduced. In Step 2, we derive a few a priori estimates and we define the pointwise limit u of the sequence (u^ε). Step 3 is devoted to the proof of the strong convergence of the sequence $T_k(u^\varepsilon)$ to $T_k(u)$ in $H_0^1(\Omega)$. In Step 4 we establish that u is a renormalized solution of (1.1).

Step 1. For an arbitrary real number $\varepsilon>0$, let us consider the approximate problem

$$\begin{cases} -\mathrm{div}\big[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big]=f & \text{in } \Omega,\\ u^\varepsilon=0 & \text{on } \partial\Omega.\end{cases} \tag{4.1}$$

Let us remark (as already mentioned in the introduction) that assumptions (H2)–(H4) do not imply that the coefficients $\mathbf{A}_{ij}(x,s)$ are bounded in $\Omega\times\mathbb{R}$. As a consequence we consider a renormalized solution u^ε of (4.1) which satisfies (see e.g. [9], [11], [12]) (for $\varepsilon>0$ fixed)

$$T_k(u^\varepsilon)\in H_0^1(\Omega),\ \forall k\geq 0; \tag{4.2}$$

$$\lim_{n\to\infty}\int_{\{n<|u^\varepsilon|<n+1\}} \big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big)\cdot Du^\varepsilon\,\mathrm{d}x=0; \tag{4.3}$$

for any function $h\in W^{1,\infty}(\mathbb{R})$ such that $\mathrm{supp}(h)$ is compact

$$\begin{aligned} &-\mathrm{div}\big[h(u^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big)\big]\\ &\qquad+h'(u^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big)\cdot Du^\varepsilon=fh(u^\varepsilon)\quad \text{in } \mathscr{D}'(\Omega).\end{aligned} \tag{4.4}$$

In (4.4) the equation takes place in $H^{-1}(\Omega)+L^1(\Omega)$ so that any test function $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ is admissible.

Indeed the existence of a solution u^ε of (4.1) satisfying (4.2)–(4.4) relies to the coercivity property $\big(\mathbf{A}(x,s)\xi+\varepsilon\xi\big)\cdot\xi\geq\varepsilon|\xi|^2$, $\forall s\in\mathbb{R}$, $\forall\xi\in\mathbb{R}^N$ (again we refer to [9], [11] and [12]). Remark that the uniqueness of u^ε is not insured without more restrictive assumption on the dependence of $\mathbf{A}(x,s)$ with respect to s (see e.g. [5] and [14]).

Step 2. In this step we derive a few a priori estimates and we define the pointwise limit of (u^ε).

Let us take $h = h_n$ in (4.4) with $T_k(u^\varepsilon)$ as a test function. Letting n tends to $+\infty$ (ε and k being fixed) and using (4.3) leads to

$$\int_\Omega \mathbf{A}(x,u^\varepsilon)Du^\varepsilon \cdot DT_k(u^\varepsilon)\,\mathrm{d}x \le k\|f\|_{L^1(\Omega)}. \tag{4.5}$$

In view of (H3), we obtain from (4.5)

$$\inf_{|s|\le k}\beta(s)\int_\Omega |DT_k(u^\varepsilon)|^2\,\mathrm{d}x \le k\|f\|_{L^1(\Omega)}, \tag{4.6}$$

so that for any $k \ge 0$

$$T_k(u^\varepsilon) \text{ is bounded in } H_0^1(\Omega). \tag{4.7}$$

From (4.7) (see [4], [9]), we deduce that there exists a measurable function u defined on Ω with values in $\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ such that for a subsequence of u^ε (still indexed by ε)

$$u^\varepsilon \longrightarrow u \quad \text{almost everywhere in } \Omega \text{ as } \varepsilon \to 0. \tag{4.8}$$

In view of (4.7)–(4.8), we have (again for a subsequence)

$$T_k(u^\varepsilon) \rightharpoonup T_k(u) \quad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{4.9}$$

Remark 4.2. Due to the properties of β ($\beta > 0$) and with the help of Poincaré inequality (see (2.1)) (4.6) implies that

$$\operatorname{meas}\{x\in\Omega\,;\,|u(x)|>k\} \le k^{-1}\Big(\inf_{|s|\le k}\beta(s)\Big)^{-1}$$

Since the condition $\int_{-\infty}^{+\infty}\beta(s)\,\mathrm{d}s < +\infty$ implies that $k^{-1}\big(\inf_{|s|\le k}\beta(s)\big)^{-1} \ge C$, we cannot deduce from (4.6) that $\operatorname{meas}\{x\in\Omega\,;\,|u(x)|=+\infty\}=0$ (as it is the case when $\beta(s)\ge\beta_0>0\ \forall s\in\mathbb{R}$!).

We consider the following two sequences

$$v^\varepsilon = \int_0^{u^\varepsilon}\gamma(s)\,\mathrm{d}s + \varepsilon u^\varepsilon, \tag{4.10}$$

$$w^\varepsilon = \int_0^{u^\varepsilon}\beta(s)\,\mathrm{d}s + \varepsilon u^\varepsilon, \tag{4.11}$$

for $\varepsilon > 0$.

Let us note that for any fixed $\varepsilon > 0$ and any fixed $k \ge 0$, there exists a real number $k_0 = k_0(\varepsilon,k) \ge 0$ such that

$$T_k(v^\varepsilon) = T_k\Big(\int_0^{T_{k_0}(u^\varepsilon)}\gamma(s)\,\mathrm{d}s + \varepsilon T_{k_0}(u^\varepsilon)\Big)$$
$$\text{and}\quad T_k(w^\varepsilon) = T_k\Big(\int_0^{T_{k_0}(u^\varepsilon)}\beta(s)\,\mathrm{d}s + \varepsilon T_{k_0}(u^\varepsilon)\Big)$$

which imply that $T_k(v^\varepsilon)$ and $T_k(w^\varepsilon)$ both belong to $H_0^1(\Omega)$ in view of (4.2). Moreover we have

$$DT_k(v^\varepsilon) = \big(\gamma(T_{k_0}(u^\varepsilon))DT_{k_0}(u^\varepsilon) + \varepsilon DT_{k_0}(u^\varepsilon)\big)T_k'(v^\varepsilon) \tag{4.12}$$

and

$$DT_k(w^\varepsilon) = \big(\beta(T_{k_0}(u^\varepsilon))DT_{k_0}(u^\varepsilon) + \varepsilon DT_{k_0}(u^\varepsilon)\big)T_k'(w^\varepsilon). \tag{4.13}$$

Using first the admissible test function $T_k(w^\varepsilon)$ in (4.4) with $h = h_n$ and letting n tends to $+\infty$ (k and ε being fixed) with the help of (4.3) leads to

$$\int_\Omega \big[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big]\cdot DT_k(w^\varepsilon)\,\mathrm{d}x = \int_\Omega f\,T_k(w^\varepsilon)\,\mathrm{d}x \tag{4.14}$$

for any $\varepsilon > 0$ and any $k \geq 0$ because of (4.13) and of the fact that $h_n(u^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big)\cdot DT_k(w^\varepsilon) = \big[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big]\cdot\big[\beta(T_{k_0}(u^\varepsilon))DT_{k_0}(u^\varepsilon) + \varepsilon DT_{k_0}(u^\varepsilon)\big]T'_k(w^\varepsilon)$ as soon as $n > k_0$. Indeed in (4.13) we have also used the fact that $fT_k(w^\varepsilon)h_n(u^\varepsilon) \to fT_k(w^\varepsilon)$ as n tends to $+\infty$ (ε and k being fixed).

Using now assumption (H3) and (4.13) in (4.14) permits us to obtain

$$\int_\Omega |DT_k(w^\varepsilon)|^2 \,\mathrm{d}x \leq k\|f\|_{L^1(\Omega)}. \tag{4.15}$$

It is well known (see e.g. [4], [8], [9]) that (4.15) implies that there exists a subsequence of w^ε, still indexed by ε, and a measurable function w defined on Ω with values in $\mathbb{R}$ (i.e. w is finite almost everywhere in Ω) such that

$$w^\varepsilon \to w \quad \text{almost everywhere in } \Omega. \tag{4.16}$$

Remark that we also deduce from (4.15) that w^ε is bounded in $W_0^{1,q}(\Omega)$ for any $1 \leq q < N/(N-1)$ so that in (4.16) we can also assume weak convergence in $W_0^{1,q}(\Omega)$ but we will not use this fact in the following.

Using (4.8), (4.10), (4.11) and (4.16) leads to

$$v^\varepsilon \to v \quad \text{almost everywhere in } \Omega \tag{4.17}$$

where $v = \int_0^u (\gamma(s) - \beta(s))\,\mathrm{d}s + w$ is measurable on Ω and finite almost everywhere in Ω because of the assumption $\int_{-\infty}^{+\infty} \gamma(s)\,\mathrm{d}s < \infty$ (see (H4)).

Now we use $T_k(v^\varepsilon)$ as a test function in (4.4) with $h = h_n$, we let n tends to $+\infty$ with the help of (4.3) to obtain

$$\int_\Omega \big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big)\cdot DT_k(v^\varepsilon)\,\mathrm{d}x = \int_\Omega fT_k(v^\varepsilon)\,\mathrm{d}x, \tag{4.18}$$

for the same reasons that the ones that lead to (4.14). Remark that in (4.18) (as in (4.14)) Du^ε has to be understood as $DT_{k_0}(u^\varepsilon)$.

Using assumption (H4) and (4.12) in (4.18) allows to obtain for any $\varepsilon > 0$ and any $k \geq 0$

$$\int_\Omega |\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon|^2 T'_k(v^\varepsilon)\,\mathrm{d}x \leq k\|f\|_{L^1(\Omega)} \tag{4.19}$$

so that for any fixed $k \geq 0$

$$\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big)T'_k(v^\varepsilon) \quad \text{is bounded in } (L^2(\Omega))^N \tag{4.20}$$

uniformly in ε (again in (4.19)–(4.20), Du^ε is identified with $DT_{k_0}(u^\varepsilon)$).

Let us end this step with an estimate which shows that the energy asymptotically vanishes on the subset where $|v^\varepsilon|$ is large (this is actually a consequence of (4.17)). For $p \geq 0$ fixed, we plug $T_{p+1}(v^\varepsilon) - T_p(v^\varepsilon)$ as a test function in (4.4) with $h = h_n$ (and use again (4.3) to let n tends to $+\infty$). It yields

$$\int_\Omega \big[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big]\cdot D\big(T_{p+1}(v^\varepsilon) - T_p(v^\varepsilon)\big)\,\mathrm{d}x = \int_\Omega f\big(T_{p+1}(v^\varepsilon) - T_p(v^\varepsilon)\big)\,\mathrm{d}x, \tag{4.21}$$

for any $\varepsilon > 0$ and any $p \geq 0$. Again in (4.21), the function u^ε is truncated because the integrand in the left hand side is 0 if $|v^\varepsilon| > p+1$. With the help of (4.17) and since v is finite almost everywhere in Ω, letting ε tends to 0 and then p tends to $+\infty$ in (4.21) gives

$$\lim_{p\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega \big[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon\big]\cdot D\big(T_{p+1}(v^\varepsilon) - T_p(v^\varepsilon)\big)\,\mathrm{d}x = 0. \tag{4.22}$$

Step 3. In this step we prove that the convergence in (4.9) is strong in $H_0^1(\Omega)$ which is the standard essential argument that allows to prove that u satisfies (2.4).

Let h be a Lipschitz continuous function on $\mathbb{R}$ with compact support. We consider equation (2.4) for u^ε with $h = h_n$ $(n \geq 1)$ and we plug the admissible test function $h_p(v^\varepsilon)T_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, for $p \geq 0$ and $k \geq 0$. It yields, as usual letting n tends to $+\infty$ and using (4.3)

$$\begin{multline}\int_\Omega T_k(u)\big(\mathbf{A}(x,u^\varepsilon)+\varepsilon Du^\varepsilon\big)\cdot Dh_p(v^\varepsilon)\,\mathrm{d}x \\ + \int_\Omega h_p(v^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)+\varepsilon Du^\varepsilon\big)\cdot DT_k(u)\,\mathrm{d}x = \int_\Omega f h_p(v^\varepsilon)T_k(u)\,\mathrm{d}x. \tag{4.23}\end{multline}$$

Indeed in (4.23), Du^ε means $DT_{k_0}(u^\varepsilon)$ where $k_0(\varepsilon, p+1)$ (because again $\mathrm{supp}(h_p) \subset [-(p+1), p+1]$ and this is the reason why one can take the limit as n tends to $+\infty$).

Now we take the limit in (4.23) as ε tends to 0. To this end we first use the fact that $\mathrm{supp}(h_p) \subset [-(p+1), p+1]$ and estimate (4.20) to extract a subsequence, still indexed by ε, such that for any $p \geq 0$

$$h_p(v^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big) \rightharpoonup X_p \quad \text{weakly in } (L^2(\Omega))^N \tag{4.24}$$

as ε tends to 0.

Then (4.17), (4.23) and (4.24) lead to

$$\begin{multline}\int_\Omega X_p\cdot DT_k(u)\,\mathrm{d}x \\ + \lim_{\varepsilon\to 0}\int_\Omega T_k(u)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big)Dh_p(v^\varepsilon)\,\mathrm{d}x = \int_\Omega f h_p(v)T_k(u)\,\mathrm{d}x \tag{4.25}\end{multline}$$

for any $p \geq 0$.

Now we prove that, for any $p \geq 0$,

$$X_p = h_p(v)\mathbf{A}(x,u)Du \quad \text{a.e. on the subset } \{x\in\Omega;\, |u(x)| < +\infty\}. \tag{4.26}$$

To establish (4.26) let us consider a continuous function h defined on $\mathbb{R}$ with compact support and let l be a real number such that $\mathrm{supp}(h) \subset [-l, l]$. Due to (4.8), (4.17) and (4.24) we have on the one hand

$$h(u^\varepsilon)h_p(v^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big) \rightharpoonup h(u)X_p \tag{4.27}$$

weakly in $(L^2(\Omega))^N$ as ε tends to 0.

On the other hand, assumption (H2), (4.8), (4.9) and (4.17) permit to obtain

$$h(u^\varepsilon)h_p(v^\varepsilon)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big) \rightharpoonup h(u)h_p(v)\mathbf{A}(x,T_l(u))DT_l(u) \tag{4.28}$$

weakly in $(L^2(\Omega))^N$ as ε tends to 0.

In view of (4.27)–(4.28) it follows that

$$h(u)X_p = h(u)h_p(v)\mathbf{A}(x,T_l(u))DT_l(u) \quad \text{almost everywhere in } \Omega,$$

which in turn implies that (4.26) holds true for any $p \geq 0$ since the function h is arbitrary with compact support. Then we are able to pass to the limit as p tends to infinity in the first term in (4.25) because $h_p(v) \to 1$ almost everywhere in Ω and indeed $DT_k(u) = 0$ almost everywhere in $\{x\in\Omega;\, |u(x)| > k\}$. As far as the second term in (4.25) is concerned we make use of (4.22) and of the fact that $|T_{p+1}(t)-T_p(t)|=1-h_p(t)$, $\forall\, t\in\mathbb{R}$ to obtain

$$\lim_{p\to+\infty}\lim_{\varepsilon\to 0}\Big|\int_\Omega T_k(u)\big(\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big)\cdot Dh_p(v^\varepsilon)\,\mathrm{d}x| \\ \le k\lim_{p\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega\big[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon\big]\cdot D\big(T_{p+1}(v^\varepsilon)-T_p(v^\varepsilon)\big)\,\mathrm{d}x=0.$$

Letting p tends to infinity in (4.25) yields

$$\int_\Omega \mathbf{A}(x,T_k(u))DT_k(u)\cdot DT_k(u)\,\mathrm{d}x=\int_\Omega fT_k(u)\,\mathrm{d}x. \tag{4.29}$$

Now using $T_k(u^\varepsilon)$ as a test function in (4.4) (again with $h=h_n$ and letting n tends to $+\infty$) leads to

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}(x,T_k(u^\varepsilon))DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,\mathrm{d}x=\int_\Omega fT_k(u)\,\mathrm{d}x \tag{4.30}$$

because of (4.9).

In view of (4.29) and (4.30), we obtain that for any $k\ge 0$

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}(x,T_k(u^\varepsilon))DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,\mathrm{d}x=\int_\Omega \mathbf{A}(x,T_k(u))DT_k(u)\cdot DT_k(u)\,\mathrm{d}x.$$

From assumption (H2) and (4.9) it follows that

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}(x,T_k(u^\varepsilon))\big(DT_k(u^\varepsilon)-DT_k(u)\big)\cdot\big(DT_k(u^\varepsilon)-DT_k(u)\big)\,\mathrm{d}x=0$$

which implies that, because of (H3),

$$T_k(u^\varepsilon)\longrightarrow T_k(u)\quad\text{strongly in } H_0^1(\Omega) \tag{4.31}$$

as ε tends to 0 for any $k\ge 0$.

Step 4. In this step we conclude the proof of Theorem 4.1 by showing that u satisfies (2.2)–(2.4).

We first prove that (2.2) holds true. Let $p\ge 1$, $n\ge 1$ and $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $D\varphi=0$ almost everywhere on $\{x\in\Omega;\, u(x)=+\infty\}$. Since $T_k(v^\varepsilon)\in H_0^1(\Omega)$ for any $k>0$, taking the admissible test function $h_p(v^\varepsilon)(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon))^+\varphi$ in (4.4) with $h=h_m$ and letting m tend to $+\infty$ (with the help of (4.3)) lead to

$$\begin{aligned}\int_{\{n<u^\varepsilon<n+1\}} & h_p(v^\varepsilon)\varphi[\mathbf{A}(x,T_{n+1}(u^\varepsilon))Du^\varepsilon+\varepsilon Du^\varepsilon]\cdot Du^\varepsilon\,\mathrm{d}x\\ &+\int_\Omega h_p(v^\varepsilon)(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon))^+[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon]\cdot D\varphi\,\mathrm{d}x\\ &+\int_\Omega h_p'(v^\varepsilon)(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon))^+\varphi[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon+\varepsilon Du^\varepsilon]\cdot Dv^\varepsilon\,\mathrm{d}x\\ &=\int_\Omega fh_p(v^\varepsilon)(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon))^+\varphi\,\mathrm{d}x.\end{aligned} \tag{4.32}$$

Remark that in the above equality $h_p(v^\varepsilon)$ stands for $h_p(T_{p+1}(v^\varepsilon))$. We now pass to the limit in (4.32) first as ε goes to zero, secondly as n goes to infinity and finally as p goes to infinity.

Using (4.8), (4.17) and the fact that v is finite almost everywhere in Ω, while $(T_{n+1}(u)-T_n(u))^+\to \mathbb{1}_{\{u=+\infty\}}$ as n goes to infinity in $L^\infty(\Omega)$ weak-$*$, we have

$$\lim_{p\to+\infty}\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega fh_p(v^\varepsilon)(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon))^+\varphi\,\mathrm{d}x=\int_{\{u=+\infty\}} f\varphi\,\mathrm{d}x. \tag{4.33}$$

Recalling that $|h'_p(v^\varepsilon)|Dv^\varepsilon = D(T_{p+1}(v^\varepsilon) - T_p(v^\varepsilon))$ and since $\|(T_{n+1}(u^\varepsilon) - T_n(u^\varepsilon))^+\varphi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}$ uniformly with to respect to n and ε, (4.22) implies that

$$\lim_{p\to+\infty}\limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega h'_p(v^\varepsilon)(T_{n+1}(u^\varepsilon) - T_n(u^\varepsilon))^+\varphi[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon]\cdot Dv^\varepsilon\,\mathrm{d}x\right| = 0. \tag{4.34}$$

From (4.8) and (4.24) it follows that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega h_p(v^\varepsilon)(T_{n+1}(u^\varepsilon) - T_n(u^\varepsilon))^+[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon]\cdot D\varphi\,\mathrm{d}x = \int_{\{u=+\infty\}} X_p\cdot D\varphi\,\mathrm{d}x. \tag{4.35}$$

We now use the essential condition $D\varphi = 0$ almost everywhere on $\{x\in\Omega;\, u(x) = +\infty\}$ to obtain from (4.35)

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_{\{n<u^\varepsilon<n+1\}} h_p(v^\varepsilon)(T_{n+1}(u^\varepsilon) - T_n(u^\varepsilon))^+[\mathbf{A}(x,u^\varepsilon)Du^\varepsilon + \varepsilon Du^\varepsilon]\cdot D\varphi\,\mathrm{d}x = 0 \tag{4.36}$$

Assumption (H2), (4.8), (4.17) and (4.27) imply that

$$\lim_{\varepsilon\to 0}\int_{\{n<u^\varepsilon<n+1\}} h_p(v^\varepsilon)\varphi[\mathbf{A}(x,T_{n+1}(u^\varepsilon))Du^\varepsilon + \varepsilon Du^\varepsilon]\cdot Du^\varepsilon\,\mathrm{d}x = \int_{\{n<u<n+1\}} h_p(v)\varphi\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x.$$

Moreover from (4.8) and the definition (4.10) of v^ε we deduce that v^ε converges to $\int_0^u\gamma(s)\,\mathrm{d}s$ as ε goes to zero almost everywhere on $\{x\in\Omega;\, |u(x)|<+\infty\}$. It follows that $v = \int_0^u\gamma(s)\,\mathrm{d}s$ almost everywhere on $\{x\in\Omega;\,|u(x)|<+\infty\}$ which in turn implies that $|v|\leq\int_{-\infty}^{+\infty}\gamma(s)\,\mathrm{d}s$ almost everywhere on the subset $\{x\in\Omega;\,|u(x)|<+\infty\}$. As a consequence if $p>\int_{-\infty}^{+\infty}\gamma(s)\,\mathrm{d}s$ then $h_p(v)=1$ almost everywhere on $\{x\in\Omega;\,|u(x)|<+\infty\}$, so that

$$\lim_{\varepsilon\to 0}\int_{\{n<u^\varepsilon<n+1\}} h_p(v^\varepsilon)\varphi[\mathbf{A}(x,T_{n+1}(u^\varepsilon))Du^\varepsilon + \varepsilon Du^\varepsilon]\cdot Du^\varepsilon\,\mathrm{d}x = \int_{\{n<u<n+1\}}\varphi\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x. \tag{4.37}$$

Gathering (4.32), (4.33), (4.34), (4.36) and (4.37) yields

$$\lim_{n\to+\infty}\int_{\{n<u<n+1\}}\varphi\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x = \int_{\{u=+\infty\}} f\varphi\,\mathrm{d}x,$$

that is (2.2).

Considering the test function $h_p(v^\varepsilon)(T_{n+1}(u^\varepsilon) - T_n(u^\varepsilon))^-\varphi$ in (4.4), with $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $D\varphi=0$ almost everywhere in $\{x\in\Omega;\,u(x)=-\infty\}$ and similar arguments lead to (2.3).

To show that (2.4) hold true we pass to the limit as ε tends to 0 in (4.4). If $k\geq 0$ is such that $\mathrm{supp}(h)\subset[-k,k]$, Du^ε means $DT_k(u^\varepsilon)$ in (4.4) so that assumption (H2), (4.8), (4.9) and (4.31) imply that

$$h(u^\varepsilon)\mathbf{A}(x,T_k(u^\varepsilon))DT_k(u^\varepsilon)\longrightarrow h(u)\mathbf{A}(x,T_k(u))DT_k(u)\quad\text{strongly in }(L^2(\Omega))^N,$$
$$\big(\mathbf{A}(x,T_k(u^\varepsilon))DT_k(u^\varepsilon) + \varepsilon DT_k(u^\varepsilon)\big)\cdot Dh(T_k(u^\varepsilon))\longrightarrow\mathbf{A}(x,T_k(u))DT_k(u)\cdot Dh(T_k(u))$$

strongly in $L^1(\Omega)$,

$$fh(u^\varepsilon)\longrightarrow fh(u)\quad\text{strongly in }L^1(\Omega),$$

as ε tends to 0.

According to the meaning of (2.4) (see the comments on Definition 2.1 in Section 2.2) we conclude that u satisfies (2.4).

The proof of Theorem 4.1 is complete. □

5. A PARTIAL UNIQUENESS RESULT

Let us assume that $\exists\alpha > 0$ such that $\gamma(s) = \alpha\beta(s)$, $\forall s \in \mathbb{R}$ and let us define the nondecreasing function $\widetilde{\beta}(r) = \int_0^r \beta(s)\,\mathrm{d}s$. Let us consider any renormalized solution u of (1.1). Then assumption (H3) and condition (2.7) imply that $\widetilde{\beta}(u) \in H_0^1(\Omega)$ (consider $\widetilde{\beta}(T_k(u)) \in H_0^1(\Omega)$ and let k tend to $+\infty$) and moreover that $D\widetilde{\beta}(u) = 0$ almost everywhere on the subset $\{x \in \Omega; |u(x)| = +\infty\}$. It follows that

$$\mathbb{1}_{\{|u|<+\infty\}}\mathbf{A}(x,u)Du = \mathbb{1}_{\{|u|<+\infty\}}\frac{\mathbf{A}(x,u)}{\beta(u)}D\widetilde{\beta}(u) \quad \text{a.e. in } \Omega.$$

As a consequence equation (2.13) rewrites as

$$-\operatorname{div}\Big[\mathbb{1}_{\{|u|<+\infty\}}\frac{\mathbf{A}(x,u)}{\beta(u)}D\widetilde{\beta}(u)\Big] - \mu_u^+ + \mu_u^- = f\,\mathbb{1}_{\{|u|<+\infty\}} \quad \text{in } \mathscr{D}'(\Omega). \tag{5.1}$$

The matrix $\frac{\mathbf{A}(x,s)}{\beta(s)}$ is uniformly coercive and bounded due to (H3)–(H4) and $\gamma(s) = \alpha\beta(s)$. It is then natural to state an assumption on the matrix $\frac{\mathbf{A}(x,s)}{\beta(s)}$ with respect to $\widetilde{\beta}(s)$ (see e.g. [3], [5] and [14]) to prove the uniqueness of u. In the following we assume that

$$\exists\alpha > 0,\ \forall s \in \mathbb{R} \quad \gamma(s) = \alpha\beta(s); \tag{H6}$$

there exists $C > 0$ such that for any $s, r \in \mathbb{R}$

$$\Big|\frac{\mathbf{A}(x,s)}{\beta(s)} - \frac{\mathbf{A}(x,r)}{\beta(r)}\Big| \le C|\widetilde{\beta}(s) - \widetilde{\beta}(r)| \quad \text{almost everywhere in } \Omega. \tag{H7}$$

We are not in a position to prove such a result essentially because we do not control the subset $\{x \in \Omega; |u(x)| = +\infty\}$ with respect to the data.

In the following Proposition, we prove that if two renormalized solutions of (1.1) are infinite on the same subset of Ω then they are equal on Ω.

Proposition 5.1. *Assume that* (H1)–(H7) *hold true. Let u and v be two renormalized solutions of* (1.1). *If (up to a set of null Lebesgue measure)*

$$\{x \in \Omega;\, u(x) = +\infty\} = \{x \in \Omega;\, v(x) = +\infty\} \tag{5.2}$$

and

$$\{x \in \Omega;\, u(x) = -\infty\} = \{x \in \Omega;\, v(x) = -\infty\}, \tag{5.3}$$

then $u = v$ almost everywhere on Ω.

Proof of Proposition 5.1. We use the equations (5.1) for u and v (and denote by $\mu_u^+, \mu_u^-, \mu_v^+, \mu_v^-$ the measures respectively corresponding to u and v in Definition 2.3). Taking the difference of $(5.1)_u$ and $(5.1)_v$ we obtain using (5.2) and (5.3)

$$\begin{aligned}-\operatorname{div}\Big[\mathbb{1}_{\{|u|<+\infty\}}\Big(\frac{\mathbf{A}(x,u)}{\beta(u)}D\widetilde{\beta}(u) - \frac{\mathbf{A}(x,v)}{\beta(v)}D\widetilde{\beta}(v)\Big)\Big]& \\ - (\mu_u^+ - \mu_v^+) + (\mu_u^- - \mu_v^-) = 0 \text{ in } \mathscr{D}'(\Omega).&\end{aligned} \tag{5.4}$$

Now the usual techniques to prove that $u = v$ is to plug the test function $\frac{1}{k^2} T_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big) \in H^1_0(\Omega) \cap L^\infty(\Omega)$ in equation (5.4) and to let k tend to 0. The only novelty here is to deal with the resulting terms involving the measures μ^+_u, μ^-_u, μ^+_v and μ^-_v for which we use the essential assumption (5.2)–(5.3). Actually conditions in (2.11)–(2.12) we can take $\varphi = T_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big) \in H^1_0(\Omega) \cap L^\infty(\Omega)$ because $DT_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big) = 0$ almost everywhere on the subset $\{x \in \Omega; |u(x)| = +\infty\} = \{x \in \Omega; |v(x)| = +\infty\}$ (see again the beginning of this section). We obtain

$$\int_\Omega T_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big)\, d\mu^+_u = \int_{\{u=+\infty\}} f\, T_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big)\, dx = 0$$

because $\widetilde{\beta}(u) = \widetilde{\beta}(v) = \int_0^{+\infty} \beta(s)\, ds$ almost everywhere on $\{x \in \Omega; u(x) = +\infty\}$. The same arguments show that all the terms involving the measures in (5.4) vanishes (when multiplied by $T_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big)$). We deduce from (5.4) that

$$\frac{1}{k^2} \int_\Omega \mathbb{1}_{\{|u|<+\infty\}} \Big(\frac{\mathbf{A}(x,u)}{\beta(u)} D\widetilde{\beta}(u) - \frac{\mathbf{A}(x,v)}{\beta(v)} D\widetilde{\beta}(v) \Big) \cdot DT_k\big(\widetilde{\beta}(u) - \widetilde{\beta}(v)\big)\, dx = 0 \tag{5.5}$$

for any $k > 0$.

Once (5.5) is established the standard method of [3] applies under assumption (H6)–(H7) and leads to $\widetilde{\beta}(u) = \widetilde{\beta}(v)$ almost everywhere in Ω which in turn gives $u = v$ almost everywhere on $\Omega \setminus \{x \in \Omega; |u(x)| = +\infty\}$. This ends the proof of Proposition 5.1. □

Remark 5.2. As shown above (in the simpler case of assumption (H7)) assumptions (5.2)–(5.3) imply that, upon using standard test function (of the form $F(\varphi(u) - \varphi(v))$) in (5.4), all the terms involving the measures μ^+_u, μ^-_u, μ^+_v and μ^-_v vanish. The reader will easily convince himself that the result of Proposition 5.1 still holds true under weaker assumption on $\mathbf{A}(x,s)$ than (H7) (such weaker assumptions are proposed in [5] and [14]).

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