QUASI–LINEAR DEGENERATE ELLIPTIC PROBLEMS WITH L^1 DATA

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ABSTRACT. We prove existence and uniqueness of a solution for a class of quasi–linear problems with L^1 data. The diffusion matrix $\mathbf{A}(x, u)$ is allowed to degenerate with respect to the unknown u. We obtain uniqueness of the solution under a weak assumption on $\mathbf{A}(x, u)$ that permits to consider highly oscillating or/and increasing coefficients (with respect to u).

1. INTRODUCTION

In this paper we study a class of possibly degenerate elliptic problems of the type

$$-\operatorname{div}(\mathbf{A}(x, u)Du) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, $f \in L^1(\Omega)$ and $\mathbf{A}(x, s) : \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ is a Carathéodory function with values in the space of matrices on $\mathbb{R}$. We assume that there exists a continuous function $\alpha : \mathbb{R} \to \mathbb{R}^+$ such that $\mathbf{A}(x, s) \geq \alpha(s)I$ for any $s \in \mathbb{R}$ and almost any x in Ω, so that equation (1.1) may degenerate on the subset $\{x \in \Omega\,;\, \alpha(u(x)) = 0\}$ of Ω. Moreover the function $\alpha(s)$ may vanish when $|s| \to +\infty$ and equation (1.1) may not be uniformly elliptic for large values of $|s|$. With respect to the "extensive" literature devoted to degenerate elliptic problems that ranges from "porous medium" equations (see e.g. the references in [16] and [17]) to problems with degeneracy at infinity (see e.g. [1], [6], [11] and [15] among the most recent papers), the originality of the present paper, as far as existence results are concerned, is to consider a diffusion matrix $\mathbf{A}(x, s)$ (which is even not assumed to be symmetric). In particular, this precludes any change of unknown in equation (1.1) as in the estimates that follows from (1.1) in order to maintain both coercivity and continuity of the diffusion matrix. Moreover we just assume that the data f belongs to $L^1(\Omega)$.

In the case where the matrix $\mathbf{A}(x, s)$ is uniformly coercive (i.e. if $\alpha(s) \geq \alpha_0 > 0$ $\forall s \in \mathbb{R}$), one can prove existence of a solution of (1.1)–(1.2), without any growth condition on

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$\mathbf{A}(x,s)$ with respect to s using the notion of renormalized solution (see e.g. [14] and [18]). Denoting by T_K ($K \geq 0$) the truncation at height K, the uniform coercivity of $\mathbf{A}(x,s)$ leads to the estimate $DT_K(u) \in \big(L^2(\Omega)\big)^N$. As a consequence the field $\mathbf{A}(x,u)DT_K(u)$ belongs to $\big(L^2(\Omega)\big)^N$ whatever the growth of $\mathbf{A}(x,s)$ may be. This is enough to give a sense to a renormalized solution of (1.1)–(1.2).

The same arguments hold true if $\alpha(s)$ is strictly positive on $\mathbb{R}$ but may tend to 0 as $|s|$ tends to infinity provided that $\int_0^{+\infty}\alpha(s)\,\mathrm{d}s = \int_{-\infty}^0 \alpha(s)\,\mathrm{d}s = +\infty$ which is a condition that insures that a solution u is finite almost everywhere in Ω. As an example, one can prove existence solution of (1.1)–(1.2) for $\alpha(t) = 1/(1+|t|)^m$ for $m \leq 1$ (and $\mathbf{A}(x,s) \geq \alpha(s)I$).

In the case where the function α may vanish on $\mathbb{R}$, one cannot expect that $DT_K(u)$ belongs to $\big(L^2(\Omega)\big)^N$ but one only has, for any $K \geq 0$, $DT_K\big(\widetilde{\alpha}(u)\big) \in \big(L^2(\Omega)\big)^N$ where $\widetilde{\alpha}(s) = \int_0^s \alpha(t)\,\mathrm{d}t$ (this could be seen by plugging the test function $T_K\big(\widetilde{\alpha}(u)\big)$ in (1.1)). This is enough to define $D\widetilde{\alpha}(u)$ almost everywhere in Ω (see [4]) and then to define Du almost everywhere in Ω through setting $Du = \frac{D\widetilde{\alpha}(u)}{\alpha(u)}$ almost everywhere on the subset $\{x \in \Omega\,;\, \alpha(u)(x) \neq 0\} = \Omega_0$ and e.g. $Du = 0$ almost everywhere on $\Omega \setminus \Omega_0$. By contrast with the coercive case, the condition $DT_K\big(\widetilde{\alpha}(u)\big) \in \big(L^2(\Omega)\big)^N$ is not sufficient, in general, to give a sense to a renormalized formulation of (1.1) since there is no reason for $\mathbf{A}(x,u)DT_K(u)$ to belong to $\big(L^2(\Omega)\big)^N$. An assumption that combines both the structure of the matrix $\mathbf{A}(x,s)$ and the growth of its coefficients with respect to s is needed. We introduce in this paper such a condition, namely for any $K \geq 0$ there exists a constant A_K such that $|\mathbf{A}(x,s)| \leq A_K$ and $|\mathbf{A}(x,s)\xi|^2 \leq A_K \mathbf{A}(x,s)\xi\cdot\xi$ for any $|s| \leq K$ and any $\xi \in \mathbb{R}^N$, almost everywhere in Ω (see (2.5) and (2.6) in Section 2). Assumption (2.5) is classical. Remark that (2.6) is not *stricto sensu* a condition on the growth of $\mathbf{A}(x,s)$ with respect to s : in the usual symmetric case it is always satisfied (whatever the growth of $\mathbf{A}(x,s)$ with respect to s may be!). Indeed if $\mathbf{A}(x,s)$ is a symmetric matrix then $|\mathbf{A}(x,s)\xi|^2 \leq |\mathbf{A}(x,s)||\mathbf{A}^{1/2}(x,s)\xi|^2 \leq A_K\mathbf{A}(x,s)\xi\cdot\xi$ where $\mathbf{A}^{1/2}(x,s)$ denotes the unique symmetric matrix such that $\mathbf{A}^{1/2}(x,s)\mathbf{A}^{1/2}(x,s) = \mathbf{A}(x,s)$. As a consequence of Theorem 2.3, we obtain an existence result in the degenerate and symmetric case without any growth assumption on $\mathbf{A}(x,s)$ (with respect to s). In the general case (i.e. for non symmetric matrices) and loosely speaking, Assumption 2.6 is concerned with the growth of the antisymmetric part of $\mathbf{A}(x,s)$ with respect to the symmetric part of $\mathbf{A}(x,s)$ (see the comments on the assumptions in Subsection 2.1).

We now turn to the uniqueness problem for equation of the type (1.1). In the non degenerate case and as far as variational solutions are concerned (i.e. for $f \in H^{-1}(\Omega)$), the most significant results in this direction can be found in [2], [9] [12], [13], where, in short, the authors assume that the matrix $\mathbf{A}(x,s)$ is Lipschitz continuous with respect to s or at least exhibit a strongly controlled modulus of continuity (in [9] an extra term of the form $\mathrm{div}(\phi(x,u))$ is involved in (1.1) with $\phi(x,s)$ Lipschitz continuous with respect to s).

For merely integrable data f (which is the case in the present paper), for non degenerate problems and if a 0–order term is involved in (1.1), namely if the operator is of the form $\lambda u - \mathrm{div}(\mathbf{A}(x,u)Du)$ with $\lambda > 0$, then one can prove uniqueness of the renormalized solution under a local Lipschitz continuity assumption on $\mathbf{A}(x,s)$ with respect to s adapting e.g. the techniques developed in [3], [18] and [20] or [5] for parabolic nonlinear version of (1.1)–(1.2). In the degenerate case (i.e. when α may vanish) a 0–order term λu, with $\lambda > 0$, permits to obtain the uniqueness of the field $\widetilde{\alpha}(u)$ which, in turn, implies the uniqueness of u. Indeed such results still hold true for a 0-order term $\beta(u)$ where β is a strictly monotone function.

With respect to the results mentioned above, a natural question arises: does the uniqueness of u (in the non degenerate case) or of $\widetilde{\alpha}(u)$ (in the degenerate case) still hold true under local Lipschitz continuity of $\mathbf{A}(x,s)$ with respect to s for Problems of the type (1.1)–(1.2) (i.e. without any 0-order term) ? We do not answer this question in the present paper so that the problem still remains open. Actually we prove an uniqueness result under a fairly technical assumption $\mathbf{A}(x,s)$ (see (3.2)) which is a global condition on $\mathbb{R}$ but which allows both strong growth of $\mathbf{A}(x,s)$ and on its modulus of continuity with respect to s. As an example we obtain uniqueness of the field $\widetilde{\alpha}(u)$ for $\mathbf{A}(x,s) = \big(1 + b(x)\exp(s)\sin^2\big(\exp(s^2)\big)\big)\mathbf{B}(\mathbf{x})$ where b is a non negative function belonging to $L^\infty(\Omega)$ and $\mathbf{B}$ is a coercive and symmetric matrix lying in $(L^\infty(\Omega))^{N\times N}$. Similar conditions have been introduced independently in the recent paper [19] to prove uniqueness of the solution in the very close framework of entropy solution when the function $\alpha(s) > 0$ for any $s \in \mathbb{R}$. Our assumption (3.2) and the one used in [19], even if similar, are not equivalent. Actually in the coercive case assumption stated in [19] forces both $\mathbf{A}(x,s)$ and its modulus of continuity to grow at most like $\exp(c|s|)$ while it is not the case with (3.5) (see the example above).

The paper is organized as follows. In Section 2 we detail the assumptions that hold true in the whole paper and we give the definition of a renormalized solution of (1.1)–(1.2). Then we prove existence of a such a solution (Theorem 2.3). This section is completed by a few estimates on renormalized solutions. Section 3 is concerned with the uniqueness of a renormalized solution of (1.1)–(1.2) under the additional assumptions mentioned above.

2. Existence result

This section is organized as follows. In subsection 2.1 we detail the assumptions on $\mathbf{A}(x,s)$. In subsection 2.2 we prove an existence result (Theorem 2.3) and in 2.3 we derive a few estimates on renormalized solutions.

2.1. **Assumptions and notations.**

In the whole paper we assume that

$$f \in L^1(\Omega); \tag{2.1}$$

$\mathbf{A}(x,s) : \Omega\times\mathbb{R} \to \mathbb{R}^{N\times N}$ is a Carathéodory function such that there exists a non negative continuous function $\alpha : \mathbb{R} \to \mathbb{R}^+$ with

$$\int_{-\infty}^{0} \alpha(s)\,\mathrm{d}s = \int_{0}^{+\infty} \alpha(s)\,\mathrm{d}s = +\infty, \tag{2.2}$$

$$\mathbf{A}(x,s)\xi\cdot\xi \geq \alpha(s)|\xi|^2, \quad \forall s\in\mathbb{R},\ \forall \xi\in\mathbb{R}^N, \text{ almost everywhere in } \Omega; \tag{2.3}$$

$$\alpha(s) = 0 \text{ implies that } \mathbf{A}(x,s) = 0 \text{ almost everywhere in } \Omega. \tag{2.4}$$

Moreover, we assume that for any $K \geq 0$, there exists a constant A_K such that

$$\big|\mathbf{A}(x,s)\big| \leq A_K \quad \text{almost everywhere in } \Omega,\ \forall s\in\mathbb{R} \text{ such that } |s|\leq K, \tag{2.5}$$

and

$$\big|\mathbf{A}(x,s)\xi\big|^2 \leq A_K\mathbf{A}(x,s)\xi\cdot\xi \quad \text{almost everywhere in } \Omega,\ \forall s\in\mathbb{R} \text{ such that } |s|\leq K. \tag{2.6}$$

A few comments on assumptions (2.3), (2.4), (2.5) and (2.6) follow. Conditions (2.3) and (2.4) say that the matrix $\mathbf{A}(x,s)$ degenerates if and only if $\alpha(s) = 0$. As mentioned in the introduction, assumption (2.6) is not a condition that concerns only the growth of $\mathbf{A}(x,s)$

with respect to s but rather a coupling hypothesis between the structure of $\mathbf{A}(x,s)$ and the growth of its coefficients: (2.6) is satisfied as soon as $\mathbf{A}(x,s)$ is symmetric (see again the introduction). As far as the non symmetric case is concerned, (2.6) is also always satisfied if $\mathbf{A}(x,s)$ does not degenerate (i.e. if $\alpha(s) > 0$ for any $s \in \mathbb{R}$). It only remains to examine the case where $\mathbf{A}(x,s)$ is non symmetric and vanishes. Conditions (2.6) appears to be a constraint on the growth of the antisymmetric part of $\mathbf{A}(x,s)$ with respect to that of the symmetric part. Since one can always assume that $\mathbf{A}(x,s) = \Lambda(x,s) + \mathbf{C}(x,s)$ where $\Lambda(x,s)$ is a diagonal matrix with $\Lambda(x,s)\xi\cdot\xi \geq \alpha(s)|\xi|^2$ and where $\mathbf{C}(x,s)$ is a antisymmetric matrix, assumption (2.6) may be rewritten as

$$\left|\Lambda^2(x,s)|\xi|^2 - \mathbf{C}^2(x,s)\xi\cdot\xi\right| \leq A_K\Lambda(x,s)\xi\cdot\xi \qquad \text{a.e. in } \Omega,\ \forall |s| \leq K.$$

The following notations will be used throughout the paper. For any positive real number K, we denote by $T_K(r)$ the truncation function at height K, $T_K(r) = \min(-K, \max(r,K))$. For any integer $n \geq 1$, let us define the bounded positive functions

$$\theta_n(s) = \frac{1}{n}\big(T_{2n}(s) - T_n(s)\big), \qquad h_n(s) = 1 - \big|\theta_n(s)\big|. \tag{2.7}$$

For any positive real number a and any measurable function v on Ω, we denote by $\{|v| < a\}$ (resp. $\{|v| \leq a\}$) the measurable subset of Ω defined by $\{x \in \Omega\,;\, |v(x)| < a\}$ (resp. $\{x \in \Omega\,;\, |v(x)| \leq a\}$). Moreover $\mathbb{1}_E$ will be used to shorten the notation $\mathbb{1}_{\{x\in E\}}$ for the characteristic function of the subset E. At last we set $\widetilde{\alpha}(r) = \int_0^r \alpha(s)\,\mathrm{d}s$ which is a $\mathcal{C}^1$ non decreasing function on $\mathbb{R}$.

2.2. **Definition of a renormalized solution to** (1.1)–(1.2)**.**

As mentioned in the introduction, we use the framework of renormalized solution to solve (1.1)–(1.2).

In order to give the definition of such a solution in our setting a few preliminaries are necessary. Let us first recall that since f belongs to $L^1(\Omega)$, even formal a priori estimates are obtained by using bounded test functions in (1.1). As usual the simplest one is $T_K(u)$ and, using (2.3), it leads to $\alpha(u)\big|DT_K(u)\big|^2 \in L^1(\Omega)$ for any $K \geq 0$. Then (and still formally) one can plug $T_K(\widetilde{\alpha}(u))$ as a test function and obtain $\alpha(u)Du\cdot DT_K(\widetilde{\alpha}(u)) \in L^1(\Omega)$ for any $K \geq 0$. Indeed, as usual when degenerate equations are involved, none of the two estimates allows to show that u belongs to a Sobolev space (or even a Lebesgue space) because $\alpha(u)$ vanishes on the subset $\{x \in \Omega \;\; \alpha(u(x)) = 0\}$. By contrast, the second estimate that may be formally rewritten $T_K(\widetilde{\alpha}(u)) \in H_0^1(\Omega)$ for any $K \geq 0$, allows us to give a sense to $D\widetilde{\alpha}(u)$ almost everywhere in Ω as shown in [4]. Therefore we are in a position to define the field Du on Ω through the formula

$$Du(x) = \begin{cases} \frac{D\widetilde{\alpha}(u)}{\alpha(u)}(x) & \text{if } \alpha(u)(x) \neq 0, \\ 0 & \text{if } \alpha(u)(x) = 0. \end{cases} \tag{2.8}$$

This is the definition of Du that will be used throughout the paper (see the comments after Definition 2.1).

A renormalized solution of (1.1)–(1.2) is proposed below.

Definition 2.1. A measurable function $u : \Omega \longrightarrow \mathbb{R}$ (u is finite almost everywhere in Ω) is a renormalized solution of (1.1)–(1.2) if

$$T_K\big(\widetilde{\alpha}(u)\big) \in H_0^1(\Omega), \quad \text{for any } K \geq 0, \tag{2.9}$$

$$\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}(x,u)Du\cdot Du \in L^1(\Omega), \quad \text{for any } K \geq 0, \tag{2.10}$$

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{|\widetilde{\alpha}(u)|\leq n\}} \alpha(u)\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x = 0, \tag{2.11}$$

and if for any function $h \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} h$ is compact, u satisfies the equation

$$-\operatorname{div}\big[h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\big] + h'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\widetilde{\alpha}(u) = fh\big(\widetilde{\alpha}(u)\big) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.12}$$

Remark 2.2 (Comments on Definition 2.1)**.** As mentioned at the beginning of this section, condition (2.9) allows to define $D\widetilde{\alpha}(u)$ almost everywhere in Ω and then Du by formula (2.8). Actually one can prove that, with these definitions,

$$D\widetilde{\alpha}(u) = \alpha(u)Du \quad \text{almost everywhere in } \Omega. \tag{2.13}$$

To this end, let us define $S = \{r \in \mathbb{R}\,;\, \alpha(r) = 0\}$ and recall that Sard's Theorem implies that the measure of the subset $\widetilde{\alpha}(S)$ is equal to 0. By a result of [10], it follows that for any K, $DT_K\big(\widetilde{\alpha}(u)\big) = 0$ almost everywhere in $\{x \in \Omega\,;\, \alpha(u)(x) = 0\}$. Then the definition of $D\widetilde{\alpha}(u)$ implies that

$$D\widetilde{\alpha}(u) = 0 \quad \text{almost everywhere in } \{x \in \Omega\,;\, \alpha(u)(x) = 0\}. \tag{2.14}$$

As a consequence of (2.8) and (2.14), we obtain (2.13).

In (2.10)–(2.12) the definition of Du is given in (2.8). Actually (2.10)–(2.12) only appeal to the value of the field Du on the subset $\{x \in \Omega\,;\, \alpha(u)(x) \neq 0\}$ since $\mathbf{A}(x,u)Du = 0$ almost everywhere on $\{x \in \Omega\,;\, \alpha(u)(x) = 0\}$ because of (2.4) whatever the value of Du may be.

The condition (2.10) is classical when dealing with renormalized solution once remarking that $\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}$ may be equivalently replaced by $\mathbb{1}_{\{|u|<K'\}}$ (for a real number K' depending on K) because of (2.2).

The renormalized equation is formally obtained through pointwise multiplication of (1.1) by $h(\widetilde{\alpha}(u))$ (and not by $h(u)$ as usual because in general $h(u)$ does not belong to $H^1(\Omega)\cap L^\infty(\Omega)$). Every term is well defined in (2.12) essentially due to assumption (2.6) and condition (2.10). Denoting by $K \geq 0$ a real number such that $\operatorname{supp} h \subset [-K, K]$, there exists a real number $K' \geq 0$ such that

$$\big|h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du\big|^2 \leq \|h\|^2_{L^\infty(\mathbb{R})}\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbb{1}_{\{|u|<K'\}}\big|\mathbf{A}(x,u)Du\big|^2 \quad \text{a.e in } \Omega,$$

because of (2.2).

Then using assumption (2.6) gives

$$\big|h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du\big|^2 \leq \|h\|^2_{L^\infty(\mathbb{R})}A_{K'}\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}(x,u)Du\cdot Du \quad \text{almost everywhere in } \Omega,$$

which in turn implies with (2.10) that

$$h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du \in \big(L^2(\Omega)\big)^N. \tag{2.15}$$

Similarly the term $h'(\widetilde{\alpha}(u))\mathbf{A}(x,u)DuD\widetilde{\alpha}(u)$ is identified with $\mathbf{A}(x,u)DuDh\big(T_K(\widetilde{\alpha}(u))\big)$ which belongs to $L^1(\Omega)$ because of (2.9) and of the fact that (2.15) implies that $\mathbb{1}_{\{|\widetilde{\alpha}(u)|<M\}}\mathbf{A}(x,u)Du \in \big(L^2(\Omega)\big)^N$ for any $M \geq 0$. It follows that (2.12) has a meaning in $H^{-1}(\Omega) + L^1(\Omega)$.

As mentioned in the introduction, notice that in the possibly degenerate case investigated in this paper, the regularity (2.9)–(2.10) on the truncates of $\widetilde{\alpha}(u)$ are not sufficient to give a sense to (2.12) (if no additional assumption on $\mathbf{A}(x,s)$ is adopted). Indeed, as far as the first term in (2.12) is concerned, writing

$$h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du = h(\widetilde{\alpha}(u))\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u),$$

could lead to the assumption $\mathbb{1}_{\{|\widetilde{\alpha}(s)|<K\}}\frac{\mathbf{A}(x,s)}{\alpha(s)} \in \big(L^\infty(\Omega\times\mathbb{R})\big)^{N\times N}$ for any $K \geq 0$. It is easy to show that this assumption is stronger than (2.6). Another possibility is to use (2.10) to give a sense to $h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du$. It leads to an assumption of the type $\mathbb{1}_{\{|\widetilde{\alpha}(s)|<K\}}\frac{\mathbf{A}(x,s)}{\alpha^{1/2}(s)} \in \big(L^\infty(\Omega\times\mathbb{R})\big)^{N\times N}$ which is weaker that the one above (that comes from (2.9)) but still stronger than (2.6).

At last, condition of type (2.11) is standard in the definition of renormalized solution, and loosely speaking is devoted to balance the lack of information in the renormalized equations (as in (2.12)) on the subsets where $|\widetilde{\alpha}(u)|$ is "large". Then (2.11) is formally obtained through using $\frac{T_n(\widetilde{\alpha}(u))}{n}$ in (1.1).

Let us end these comments with examining a few situations where Definition 2.1 meets the classical notion of a weak solution of (1.1)–(1.2). Let u be a solution in the sense of Definition 2.1. First of all, it is clear that (2.3) and (2.11) imply that

$$\int_\Omega \big|DT_K(\widetilde{\alpha}(u))\big|^2\,\mathrm{d}x \leq CK,$$

for any $K \geq 0$. A classical result of [7], [8], then allows to conclude that $\widetilde{\alpha}(u)$ belongs to $W_0^{1,q}(\Omega)$ for any $1 \leq q < \frac{N}{N-1}$. Assume now that $|\mathbf{A}(x,s)| \leq C\alpha(s)$ for some positive constant C. Then

$$\big|\mathbf{A}(x,u)Du\big| \leq \alpha(u)|Du| = \big|D\widetilde{\alpha}(u)\big| \quad \text{almost everywhere in } \Omega$$

and we obtain that $\mathbf{A}(x,u)Du$ belongs to $L^q(\Omega)$ for any $1 \leq q < \frac{N}{N-1}$. Choosing $h = h_n$ in (2.12), where h_n is defined in (2.7), letting n tends to $+\infty$ and using (2.11) give

$$-\operatorname{div}\big[\mathbf{A}(x,u)Du\big] = f \quad \text{in } \mathcal{D}'(\Omega), \tag{2.16}$$

which shows that u is weak solution of (1.1) (when Du is defined through (2.8)).

Notice that equation (2.16) may be alternatively written as

$$-\operatorname{div}\Big[\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u)\Big] = f \quad \text{in } \mathcal{D}'(\Omega), \tag{2.17}$$

whatever the value of the matrix $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ is arbitrarily fixed on the subset $\{s \in \mathbb{R}\,;\, \alpha(s) = 0\}$ because (2.4) and (2.13) imply that

$$\mathbf{A}(x,u)Du = \frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u) \quad \text{almost everywhere in } \Omega.$$

One concludes that if u is a solution in the sense of Definition 2.1, then $\widetilde{\alpha}(u)$ is a weak solution of (2.17) which belongs to $W_0^{1,q}(\Omega)$. Let us point out that reducing equation (1.1) to equation (2.17) is not always possible under assumptions (2.2)–(2.6) because, in general, they do not imply that $\frac{\mathbf{A}(x,u)}{\alpha(u)} \in \big(L^\infty(\Omega)\big)^{N\times N}$.

2.3. **Existence result.**

Theorem 2.3. *Under assumptions* (2.1)–(2.5) *there exists a renormalized solution in the sense of Definition 2.1.*

Remark 2.4. The comments at the end of Subsection 2.2 and Theorem 2.3 provide the existence of a weak solution $\widetilde{\alpha}(u)$ in $W_0^{1,q}(\Omega)$ of the problem

$$-\operatorname{div}\Big[\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u)\Big]=f,\quad \text{in } \mathcal{D}'(\Omega)$$

as soon as $|\mathbf{A}(x,s)|\le C\alpha(s)$ almost everywhere in Ω and $\forall s\in\mathbb{R}$ (with arbitrary values of $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ when $\alpha(s)=0$). Let us emphasize that under the assumptions of Theorem 2.3, the matrix field $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ is not necessarily continuous with respect to $\widetilde{\alpha}(s)$ so that such an existence result for $\widetilde{\alpha}(u)$ is not straightforward (in this direction see the example (3.7) at the end of Subsection 3.1).

Proof of Theorem 2.3. The proof relies on an approximation procedure, a priori estimates and passing to the limit.
Step 1. Let f^ε be a sequence of $L^2(\Omega)$ such that

$$f^\varepsilon \longrightarrow f \quad \text{strongly in } L^1(\Omega) \text{ as } \varepsilon \text{ goes to zero,} \tag{2.18}$$

and set for any $\varepsilon>0$

$$\mathbf{A}^\varepsilon(x,s)=\mathbf{A}\big(x,T_{1/\varepsilon}(s)\big)+\varepsilon I,\qquad \text{for any } s\in\mathbb{R} \text{ and almost everywhere in } \Omega, \tag{2.19}$$

where I is the identity matrix of $\mathbb{R}^{N\times N}$.

Indeed, due to the properties of $\mathbf{A}(x,s)$, $\mathbf{A}^\varepsilon$ is a bounded field of matrices and

$$\mathbf{A}^\varepsilon(x,s)\xi\cdot\xi\ge\varepsilon|\xi|^2,\qquad \text{for any } s\in\mathbb{R}, \text{ any } \xi\in\mathbb{R}^N \text{ and almost everywhere in } \Omega. \tag{2.20}$$

As a consequence, for any $\varepsilon>0$, there exists at least a weak solution $u^\varepsilon\in H_0^1(\Omega)$ of the approximate problem

$$-\operatorname{div}\big[\mathbf{A}^\varepsilon(x,u^\varepsilon)Du^\varepsilon\big]=f^\varepsilon \quad \text{in } \Omega, \tag{2.21}$$

$$u^\varepsilon=0 \quad \text{on } \partial\Omega. \tag{2.22}$$

Step 2. In this step we derive a few a priori estimates and we identify the limit of the sequence u^ε.

For any $K\ge 0$, the function $T_K\big(\widetilde{\alpha}(s)\big)$ is indeed a bounded Lipschitz function on $\mathbb{R}$ with derivative $T_K'\big(\widetilde{\alpha}(s)\big)\alpha(s)\mathbb{1}_{\{|\widetilde{\alpha}(s)|\ne K\}}=T_K'\big(\widetilde{\alpha}(s)\big)\alpha(s)$ for almost any s in the subset $\{s\in\mathbb{R}\,;\,\widetilde{\alpha}(s)\ne K\}$ since $\widetilde{\alpha}\in\mathcal{C}^1(\mathbb{R})$. Then $T_K\big(\widetilde{\alpha}(u^\varepsilon)\big)$ is an admissible test function in (2.21)–(2.22) which leads to

$$\int_\Omega T_K'\big(\widetilde{\alpha}(u^\varepsilon)\big)\alpha(u^\varepsilon)\mathbf{A}\big(x,T_{1/\varepsilon}(u^\varepsilon)\big)Du^\varepsilon\cdot Du^\varepsilon\,\mathrm{d}x + \int_\Omega \varepsilon T_K'\big(\widetilde{\alpha}(u^\varepsilon)\big)\alpha(u^\varepsilon)|Du^\varepsilon|^2\,\mathrm{d}x=\int_\Omega f^\varepsilon T_K\big(\widetilde{\alpha}(u^\varepsilon)\big)\,\mathrm{d}x. \tag{2.23}$$

Since $T_K'\ge 0$ and $\alpha\ge 0$, we deduce from (2.18) and (2.23) that

$$\int_\Omega T_K'\big(\widetilde{\alpha}(u^\varepsilon)\big)\alpha(u^\varepsilon)\mathbf{A}\big(x,T_{1/\varepsilon}(u^\varepsilon)\big)Du^\varepsilon\cdot Du^\varepsilon\,\mathrm{d}x\le CK, \tag{2.24}$$

where C is a constant depending on $\|f\|_{L^1(\Omega)}$.

Now the assumption $\int_0^{+\infty}\alpha(s)\,\mathrm{d}s=\int_{-\infty}^0\alpha(s)\,\mathrm{d}s=+\infty$ implies that for ε small enough (K being fixed)

$$T_K'\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x,T_{1/\varepsilon}(u^\varepsilon)\big)=T_K'\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x,u^\varepsilon\big)$$

since $T_K'\big(\widetilde{\alpha}(u^\varepsilon)\big)=0$ if $\big|\widetilde{\alpha}(u^\varepsilon)\big|>K$.

Then (2.3) and (2.24) imply that for ε small enough

$$\int_\Omega\big|DT_K\big(\widetilde{\alpha}(u^\varepsilon)\big)\big|^2\,\mathrm{d}x\le CK, \tag{2.25}$$

and due to (2.22)

$$T_K\big(\widetilde{\alpha}(u^\varepsilon)\big)\quad\text{is bounded in } H_0^1(\Omega).$$

As a consequence there exists a subsequence (still indexed by ε) $\widetilde{\alpha}(u^\varepsilon)$ and a measurable function $v:\Omega\to\mathbb{R}$, finite almost everywhere in Ω, such that

$$\widetilde{\alpha}(u^\varepsilon)\longrightarrow v\quad\text{almost everywhere in }\Omega, \tag{2.26}$$

$$T_K\big(\widetilde{\alpha}(u^\varepsilon)\big)\rightharpoonup T_K(v)\quad\text{weakly in }H_0^1(\Omega) \tag{2.27}$$

as ε tends to zero.

For almost any x in Ω, let us now define the function u by

$$u(x)=\inf\big\{r\in\mathbb{R}\,;\,v(x)\le\widetilde{\alpha}(r)\big\}. \tag{2.28}$$

Remark that u is well defined since $\widetilde{\alpha}$ is non decreasing and onto on $\mathbb{R}$. Moreover the function u is measurable on Ω (since $\beta(r)=\inf\{s\in\mathbb{R}\,;\,r\le\widetilde{\alpha}(s)\}$) is an non decreasing function on $\mathbb{R}$). Indeed the definition (2.28) shows that $\widetilde{\alpha}(u)=v$ and (2.26)–(2.27) rewrites as

$$\widetilde{\alpha}(u^\varepsilon)\longrightarrow\widetilde{\alpha}(u)\quad\text{almost everywhere in }\Omega\text{ as }\varepsilon\to0 \tag{2.29}$$

$$T_K\big(\widetilde{\alpha}(u^\varepsilon)\big)\rightharpoonup T_K\big(\widetilde{\alpha}(u)\big)\quad\text{weakly in }H_0^1(\Omega)\text{ as }\varepsilon\to0. \tag{2.30}$$

Remark that (2.26)–(2.29) show that $\widetilde{\alpha}(u)$ and u are finite almost everywhere in Ω since v is finite almost everywhere in Ω and $\int_0^{+\infty}\alpha(s)\,\mathrm{d}s=\int_{-\infty}^0\alpha(s)\,\mathrm{d}s=+\infty$. Moreover (2.29) also implies that

$$u^\varepsilon\longrightarrow u\quad\text{almost everywhere in }\Omega\setminus\{x\in\Omega\,;\,\alpha(u)(x)\ne0\}, \tag{2.31}$$

since $\big(\widetilde{\alpha}\big)^{-1}$ is continuous on the subset $\widetilde{\alpha}\big(\{r\in\mathbb{R}\,;\,\alpha(r)\ne0\}\big)$. Using (2.29) it is easy to show that

$$\alpha(u^\varepsilon)\longrightarrow\alpha(u)\quad\text{almost everywhere in }\Omega, \tag{2.32}$$

because $\widetilde{\alpha}$ is an $\mathcal{C}^1$ non decreasing function.

Let us take $T_K(u^\varepsilon)$ as a test function in (2.21), it yields

$$\int_\Omega\mathbf{A}\big(x,T_{1/\varepsilon}(u^\varepsilon)\big)Du^\varepsilon\cdot DT_K(u^\varepsilon)\,\mathrm{d}x+\int_\Omega\varepsilon\big|DT_K(u^\varepsilon)\big|^2\,\mathrm{d}x=\int_\Omega f^\varepsilon T_K(u^\varepsilon)\,\mathrm{d}x.$$

Then (2.18), (2.19) give for ε small enough

$$\int_\Omega\mathbf{A}(x,u^\varepsilon)DT_K(u^\varepsilon)\cdot DT_K(u^\varepsilon)\,\mathrm{d}x+\varepsilon\int_\Omega\big|DT_K(u^\varepsilon)\big|^2dx\le CK,$$

and as a consequence of (2.3) (recall that α is a positive function) for any $K \geq 0$

$$\mathbf{A}(x, T_K(u^\varepsilon))DT_K(u^\varepsilon) \cdot DT_K(u^\varepsilon) \quad \text{is bounded in } L^1(\Omega), \tag{2.33}$$

$$\varepsilon^{1/2} DT_K(u^\varepsilon) \quad \text{is bounded in } \big(L^2(\Omega)\big)^N. \tag{2.34}$$

We now use assumption (2.6), which plays here a decisive role, to obtain together with (2.33) that for any $K \geq 0$

$$\mathbf{A}(x, T_K(u^\varepsilon))DT_K(u^\varepsilon) \text{ is bounded in } \big(L^2(\Omega)\big)^N. \tag{2.35}$$

For any integer $n \geq 1$ let us take $K = n$ in (2.23) to obtain

$$\frac{1}{n}\int_{\{|\widetilde{\alpha}(u^\varepsilon)|<n\}} \alpha(u^\varepsilon)\mathbf{A}^\varepsilon\big(x, T_{1/\varepsilon}(u^\varepsilon)\big)Du^\varepsilon \cdot Du^\varepsilon \,\mathrm{d}x \leq \frac{1}{n}\int_\Omega f^\varepsilon T_n\big(\widetilde{\alpha}(u^\varepsilon)\big)\,\mathrm{d}x.$$

Using $\int_0^{+\infty}\alpha(s)\,\mathrm{d}s = \int_{-\infty}^0 \alpha(s)\,\mathrm{d}s = +\infty$, it follows that for ε small enough

$$\frac{1}{n}\int_{\{|\widetilde{\alpha}(u^\varepsilon)|<n\}} \alpha(u^\varepsilon)\mathbf{A}^\varepsilon(x, u^\varepsilon)Du^\varepsilon \cdot Du^\varepsilon \,\mathrm{d}x \leq \frac{1}{n}\int_\Omega f^\varepsilon T_n\big(\widetilde{\alpha}(u^\varepsilon)\big)\,\mathrm{d}x, \tag{2.36}$$

for any $n \geq 1$. Due to (2.18), (2.29), taking the limit-sup as ε tends to zero in (2.36) gives

$$\overline{\lim_{\varepsilon\to 0}}\,\frac{1}{n}\int_{\{|\widetilde{\alpha}(u^\varepsilon)|<n\}} \alpha(u^\varepsilon)\mathbf{A}^\varepsilon(x, u^\varepsilon)Du^\varepsilon \cdot Du^\varepsilon \,\mathrm{d}x \leq \frac{1}{n}\int_\Omega f T_n\big(\widetilde{\alpha}(u)\big)\,\mathrm{d}x,$$

for any $n \geq 1$. The function $\widetilde{\alpha}(u)$ being finite almost everywhere in Ω, the sequence $\frac{1}{n}T_n\big(\widetilde{\alpha}(u)\big)$ converges to 0 almost everywhere in Ω as n tends to $+\infty$ and is bounded by 1. Lebesgue convergence theorem allows us to conclude that

$$\lim_{n\to+\infty}\overline{\lim_{\varepsilon\to 0}}\,\frac{1}{n}\int_{\{|\widetilde{\alpha}(u^\varepsilon)|<n\}} \alpha(u^\varepsilon)\mathbf{A}^\varepsilon(x, u^\varepsilon)Du^\varepsilon \cdot Du^\varepsilon \,\mathrm{d}x = 0, \tag{2.37}$$

Step 3. In this step we prove the following lemma.

Lemma 2.5. *For a subsequence, still indexed by ε and for any function $h \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} h$ is compact, we have*

$$h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}^\varepsilon(x, u^\varepsilon)Du^\varepsilon \rightharpoonup h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x, u)Du \quad \text{weakly in } (L^2(\Omega))^N, \text{ as } \varepsilon \text{ tends to zero.} \tag{2.38}$$

Remark 2.6. In (2.38) as in Definition 2.1, Du is defined by (2.8).

Proof of Lemma 2.5. Let h be in $W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} h$ compact and denote by K a positive real number such that $\operatorname{supp} h \subset [-K, K]$. Due to the conditions $\int_0^{+\infty}\alpha(s)\,\mathrm{d}s = \int_{-\infty}^0 \alpha(s)\,\mathrm{d}s = +\infty$, there exists a positive real number K' such that $\{x \in \Omega\,;\, |\widetilde{\alpha}(u)(x)| \leq K\} \subset \{x \in \Omega\,;\, |u(x)| \leq K'\}$. It follows that

$$\begin{aligned} h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}^\varepsilon(x, u^\varepsilon)Du^\varepsilon &= h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}^\varepsilon\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon) \\ &= h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon) + \varepsilon h\big(\widetilde{\alpha}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon) \end{aligned} \tag{2.39}$$

for $\frac{1}{\varepsilon} > K'$, because of the definition (2.19) of $\mathbf{A}^\varepsilon(x, s)$.

Estimates (2.34)–(2.35) show that $h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}^\varepsilon(x, u^\varepsilon)Du^\varepsilon$ is bounded in $\big(L^2(\Omega)\big)^N$. To the expense of extracting subsequences and since $\varepsilon h\big(\widetilde{\alpha}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon)$ indeed converges to 0 strongly in $(L^2(\Omega))^N$, establishing (2.38) reduces to show that

$$h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon) \rightharpoonup h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x, u)Du \quad \text{weakly in } (L^2(\Omega))^N, \tag{2.40}$$

as ε tend to 0.

To prove that (2.40) holds true we write for any $\delta > 0$

$$h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon) = \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta} h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon) + \Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big) h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon).$$

Then we prove that

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}} \int_\Omega \Big|\Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big) h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon)\Big|^2 \,\mathrm{d}x = 0 \tag{2.41}$$

and

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_\Omega \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta} h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon)\cdot\psi\,\mathrm{d}x = \int_\Omega h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)DT_{K'}(u)\cdot\psi\,\mathrm{d}x \tag{2.42}$$

for any $\psi \in \big(L^2(\Omega)\big)^N$, which implies that (2.40) holds true.

As far as (2.41) is concerned, let us set for $t \geq 0$

$$z_\delta(t) = \begin{cases} 0 & \text{if } t = 0,\\ 1 - \frac{T_\delta(t)}{\delta} & \text{if } t > 0. \end{cases}$$

Then due to the fact that $\mathbf{A}(x,t) = 0$ if $\alpha(t) = 0$ (see (2.4)), we have for any $0 < \varepsilon < \frac{1}{K'}$ and $\delta > 0$

$$\Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) = z_\delta\big(\alpha(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) \quad \text{almost everywhere in } \Omega.$$

For any $\delta > 0$, the function $\Psi_\delta(t) = \int_0^t z_\delta\big(\alpha(s)\big)\,\mathrm{d}s$ is indeed an increasing Lipschitz continuous function defined on $\mathbb{R}$. Using $\Psi_\delta\big(T_{K'}(u^\varepsilon)\big)$ as a test function in (2.21) leads to for $0 < \varepsilon < \frac{1}{K'}$

$$0 \leq \int_\Omega \Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big)\mathbf{A}^\varepsilon\big(x, T_{K'}(u^\varepsilon)\big)Du^\varepsilon\cdot DT_{K'}(u^\varepsilon)\,\mathrm{d}x = \int_\Omega f^\varepsilon\Psi_\delta\big(T_{K'}(u^\varepsilon)\big)\,\mathrm{d}x.$$

Since Ψ_δ is increasing and using (2.18), it follows that

$$0 \leq \overline{\lim_{\varepsilon\to 0}} \int_\Omega \Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon)\cdot DT_{K'}(u^\varepsilon)\,\mathrm{d}x \leq C\max\big(\Psi_\delta(K'), |\Psi_\delta(-K')|\big), \tag{2.43}$$

where C is a constant depending on $\|f\|_{L^1(\Omega)}$.

Now the function $z_\delta\big(\alpha(s)\big)$ converges to 0 for any $s \in \mathbb{R}$ as δ tends to 0 (remark here that we use $z_\delta(t) = 0$ for $t = 0$) so that Lebesgue convergence theorem implies that $\Psi_\delta(K')$ converges to 0 as δ tends to 0. Similarly $\Psi_\delta(-K')$ converges to 0 as δ tends to 0 and we conclude from (2.43) that

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}} \int_\Omega \Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon)\cdot DT_{K'}(u^\varepsilon)\,\mathrm{d}x = 0.$$

Now, using (2.6) we have

$$\int_\Omega \left|\Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big) h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon)\right|^2 \mathrm{d}x$$
$$\leq \|h\|^2_{L^\infty(\mathbb{R})} A_{K'} \int_\Omega \Big(1 - \frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\Big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon) \cdot DT_{K'}(u^\varepsilon)\,\mathrm{d}x,$$

which implies that (2.41) holds true.

In order to establish (2.42), we investigate separately the behavior of the integrand on the subset $\Omega_1 = \{x \in \Omega\,;\, \alpha(u)(x) = 0\}$ and on $\Omega \setminus \Omega_1$.

As far as Ω_1 is concerned, we write

$$\left| h\big(\widetilde{\alpha}(u^\varepsilon)\big)\frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\psi\right| \leq \frac{C_{K'}}{\delta}\alpha(u^\varepsilon)|\psi| \quad \text{almost everywhere in } \Omega \tag{2.44}$$

for any $\varepsilon > 0$ and $\delta > 0$ and where $C_{K'}$ is a constant independent of ε and δ.

Since $\alpha(u^\varepsilon) \to 0$ on Ω_1 (see (2.32)), the sequence of left-hand side of (2.44) tends to 0 almost everywhere in Ω_1 as ε tends to 0 (K' and δ being kept fixed). Indeed this sequence is uniformly bounded with respect to ε so that Lebesgue convergence theorem implies that

$$h\big(\widetilde{\alpha}(u^\varepsilon)\big)\frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\psi \longrightarrow 0 \quad \text{in } \big(L^2(\Omega_1)\big)^N \text{ as } \varepsilon \text{ tends to } 0. \tag{2.45}$$

Now estimate (2.35) and (2.45) permit to conclude that

$$\int_{\Omega_1} h\big(\widetilde{\alpha}(u^\varepsilon)\big)\frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon) \cdot \psi\,\mathrm{d}x \longrightarrow 0 \tag{2.46}$$

as ε tends to 0 for any $\psi \in \big(L^2(\Omega)\big)^N$.

As far as $\Omega \setminus \Omega_1$ is concerned, estimate (2.35) implies that (for a subsequence still indexed by ε)

$$\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon) \rightharpoonup w_{K'} \quad \text{weakly in } \big(L^2(\Omega)\big)^N \tag{2.47}$$

as ε tends to 0 where $w_{K'} \in \big(L^2(\Omega)\big)^N$.

The pointwise convergence of u^ε to u in $\Omega \setminus \Omega_1$ (see (2.41)) then allows us to show that $h\big(\widetilde{\alpha}(u)\big) w_{K'} = h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x, u) Du$ almost everywhere in $\Omega \setminus \Omega_1$ where Du is defined in (2.8). Indeed we have on the one hand

$$\alpha(u^\varepsilon) h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon) \rightharpoonup \alpha(u) h\big(\widetilde{\alpha}(u)\big) w_{K'} \quad \text{weakly in } \big(L^2(\Omega)\big)^N \tag{2.48}$$

as ε tends to 0, because of (2.29) and (2.32).

On the other hand due to the choice of K and K'

$$\alpha(u^\varepsilon) h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon) = h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_K\big(\widetilde{\alpha}(u^\varepsilon)\big)$$

and (2.29), (2.30), (2.31) together with the properties of the matrix $\mathbf{A}$ ($\mathbf{A}$ is a Carathéodory function and verifies (2.5)) imply that

$$\alpha(u^\varepsilon) h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x, T_{K'}(u^\varepsilon)\big) DT_{K'}(u^\varepsilon)$$
$$\rightharpoonup h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x, T_{K'}(u)) DT_K\big(\widetilde{\alpha}(u)\big) \quad \text{weakly in } \big(L^2(\Omega \setminus \Omega_1)\big)^N \tag{2.49}$$

as ε tends to 0.

Comparing (2.48) and (2.49) we deduce that

$$\begin{aligned}\alpha(u)h\big(\widetilde{\alpha}(u)\big)w_{K'} &= h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,T_{K'}(u))DT_K\big(\widetilde{\alpha}(u)\big)\\ &= \mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}(x,T_{K'}(u))D\widetilde{\alpha}(u) \quad \text{almost everywhere in } \Omega\setminus\Omega_1,\end{aligned}\tag{2.50}$$

which, together with the definition (2.8) of Du, permit to conclude that

$$h\big(\widetilde{\alpha}(u)\big)w_{K'} = h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du \quad \text{almost everywhere in } \Omega\setminus\Omega_1\tag{2.51}$$

because $\alpha(u)(x)\neq 0$ on $\Omega\setminus\Omega_1$ and the choice of K and K'.

Gathering (2.47) and (2.51), we finally obtain (using also (2.29) and (2.32))

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_1}&\frac{T_\delta\big(\alpha(u^\varepsilon)\big)}{\delta}h\big(\widetilde{\alpha}(u^\varepsilon)\big)\mathbf{A}\big(x,T_{K'}(u^\varepsilon)\big)DT_{K'}(u^\varepsilon)\cdot\psi\,\mathrm{d}x\\ &= \int_{\Omega\setminus\Omega_1}\frac{T_\delta\big(\alpha(u)\big)}{\delta}h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot\psi\,\mathrm{d}x\\ &= \int_{\Omega}\frac{T_\delta\big(\alpha(u)\big)}{\delta}h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot\psi\,\mathrm{d}x\end{aligned}\tag{2.52}$$

for any $\psi\in\big(L^2(\Omega)\big)^N$; the last equality being due to $T_\delta(\alpha(u))=0$ almost everywhere on Ω_1.

In (2.52), passing to the limit as δ tends to 0 is an easy task remarking that $\mathbf{A}(x,u)=0$ on the subset $\{x\in\Omega\,;\,\alpha(u)(x)=0\}$ (due to (2.4)–(2.5)) so that

$$h\big(\widetilde{\alpha}(u)\big)\frac{T_\delta\big(\alpha(u)\big)}{\delta}\mathbf{A}(x,u)\longrightarrow h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u) \quad \text{almost everywhere in } \Omega$$

as δ tends to 0. Moreover the integrand of the right hand-side of (2.52) is bounded by the L^1-function $h\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot\psi$ and Lebesgue convergence theorem permits to pass to the limit as δ tends to zero in (2.52) and to obtain (2.42). This concludes the proof of Lemma 2.5. □

Step 4. In this step we prove that u is a renormalized solution of (1.1)–(1.2).

From (2.30) it follows that $T_K\big(\widetilde{\alpha}(u)\big)$ belongs to $H_0^1(\Omega)$ for any $K\geq 0$ and then u verifies (2.9).

Let K be a positive real number and let $\mathbf{A}_s$ be the symmetric part of the matrix $\mathbf{A}$. The techniques already used in the proof of Lemma 2.5 allow us to show that

$$\mathbb{1}_{\{|\widetilde{\alpha}(u^\varepsilon)|<K\}}\mathbf{A}_s^{1/2}(x,u^\varepsilon)Du^\varepsilon\rightharpoonup\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}_s^{1/2}(x,u)Du \quad \text{weakly in } \big(L^2(\Omega\setminus\Omega_1)\big)^N\tag{2.53}$$

as ε goes to zero and where $\Omega_1=\{x\in\Omega\,;\,\alpha(u)(x)=0\}$. Indeed from (2.33), the field $\mathbb{1}_{\{|\widetilde{\alpha}(u^\varepsilon)|<K\}}\mathbf{A}_s^{1/2}(x,u^\varepsilon)Du^\varepsilon$ is bounded in $\big(L^2(\Omega)\big)^N$ and, up to extracting subsequences, it converges weakly to z_K in $\big(L^2(\Omega)\big)^N$, where z_K belongs to $\big(L^2(\Omega)\big)^N$. Let K' be a positive real number such that $\{x\in\Omega\,;\,|\widetilde{\alpha}(u)(x)|\leq K\}\subset\{x\in\Omega\,;\,|u(x)|\leq K'\}$. Therefore we have

$$\begin{aligned}\alpha(T_{K'}(u^\varepsilon))\mathbb{1}_{\{|\widetilde{\alpha}(u^\varepsilon)|<K\}}\mathbf{A}_s^{1/2}(x,u^\varepsilon)Du^\varepsilon &= \alpha(u^\varepsilon)\mathbb{1}_{\{|\widetilde{\alpha}(u^\varepsilon)|<K\}}\mathbf{A}_s^{1/2}(x,u^\varepsilon)Du^\varepsilon\\ &= \mathbf{A}_s^{1/2}(x,T_{K'}(u^\varepsilon))DT_K\big(\widetilde{\alpha}(u^\varepsilon)\big).\end{aligned}$$

From (2.29), (2.30) and (2.31) together with the properties of the matrix $\mathbf{A}$, by identifying the weak limit in $\big(L^2(\Omega)\big)^N$ of each term in the above equality as ε goes to zero we obtain that $\alpha(T_{K'}(u))z_K=\mathbf{A}_s^{1/2}(x,u)DT_K\big(\widetilde{\alpha}(u)\big)$ almost everywhere in Ω. From the definition (2.8) of Du it follows that $\alpha(T_{K'}(u))z_K = \alpha(u)\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}_s^{1/2}(x,u)Du$ almost everywhere. Since $\alpha(u) \neq 0$ almost everywhere in $\Omega \setminus \Omega_1$ we conclude that $z_K = \mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}_s^{1/2}(x,u)Du$ almost everywhere in Ω_1 and then (2.53) holds true.

Recalling that $\mathbf{A}(x,u) = 0$ almost everywhere in Ω_1, the field $\mathbb{1}_{\{|\widetilde{\alpha}(u)|<K\}}\mathbf{A}_s^{1/2}(x,u)Du$ belongs to $\left(L^2(\Omega)\right)^N$, that is (2.10).

From (2.32) and (2.53) we have, for any $n \geq 1$,

$$\mathbb{1}_{\{|\widetilde{\alpha}(u^\varepsilon)|<n\}}\alpha^{1/2}(u^\varepsilon)\mathbf{A}_s^{1/2}(x,u^\varepsilon)Du^\varepsilon \rightharpoonup \mathbb{1}_{\{|\widetilde{\alpha}(u)|<n\}}\alpha^{1/2}(u)\mathbf{A}_s^{1/2}(x,u)Du$$

weakly in $\left(L^2(\Omega \setminus \Omega_1)\right)^N$ as ε goes to zero. Since $\mathbf{A}(x,u) = 0$ almost everywhere in Ω_1, from (2.19) and (2.37) it follows that

$$\lim_{n\to+\infty} \frac{1}{n}\int_{\{|\widetilde{\alpha}(u)|\leq n\}} \alpha(u)\mathbf{A}(x,u)Du \cdot Du \,\mathrm{d}x = 0$$

Now we claim that (2.12) holds true. Let $h \in W^{1,\infty}(\mathbb{R})$ such that $\mathrm{supp}(h)$ is compact and let $\varphi \in \mathcal{C}_0^\infty(\Omega)$. For any $n \geq 1$ the function $h_n(\widetilde{\alpha}(u^\varepsilon))h(\widetilde{\alpha}(u))\varphi$ belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$, and then it is an admissible test function in (2.21). It yields that, for any $n \geq 1$,

$$\begin{aligned} \text{(2.54)} \quad & \int_\Omega h_n(\widetilde{\alpha}(u^\varepsilon))\mathbf{A}^\varepsilon(x,u^\varepsilon)Du^\varepsilon \cdot D[h(\widetilde{\alpha}(u))\varphi]\,\mathrm{d}x \\ & + \int_\Omega \alpha(u^\varepsilon)h_n'(\widetilde{\alpha}(u^\varepsilon))h(\widetilde{\alpha}(u))\varphi\mathbf{A}^\varepsilon(x,u^\varepsilon)Du^\varepsilon \cdot Du^\varepsilon\,\mathrm{d}x = \int_\Omega f^\varepsilon h_n(\widetilde{\alpha}(u^\varepsilon))h(\widetilde{\alpha}(u))\varphi\,\mathrm{d}x. \end{aligned}$$

Let us pass to limit in (2.54) as ε goes to zero and as n goes to infinity.

The field $h(\widetilde{\alpha}(u))\varphi$ being bounded in $L^\infty(\Omega)$, the definition of h_n ($h_n'(r) = \mathbb{1}_{\{n<|r|<2n\}}\mathrm{sign(r)}/n$ almost everywhere in $\mathbb{R}$) together with (2.37) allows us to obtain

$$\text{(2.55)} \qquad \lim_{n\to+\infty}\overline{\lim_{\varepsilon\to 0}}\int_\Omega \alpha(u^\varepsilon)h_n'(\widetilde{\alpha}(u^\varepsilon))h(\widetilde{\alpha}(u))\varphi\mathbf{A}^\varepsilon(x,u^\varepsilon)Du^\varepsilon \cdot Du^\varepsilon\,\mathrm{d}x = 0.$$

The function $h(\widetilde{\alpha}(u))\varphi$ belonging to $H_0^1(\Omega)$, from Lemma 2.5 we have

$$\lim_{\varepsilon\to 0}\int_\Omega h_n(\widetilde{\alpha}(u^\varepsilon))\mathbf{A}^\varepsilon(x,u^\varepsilon)Du^\varepsilon \cdot D[h(\widetilde{\alpha}(u))\varphi]\,\mathrm{d}x = \int_\Omega h_n(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du \cdot D[h(\widetilde{\alpha}(u))\varphi]\,\mathrm{d}x.$$

Since $D[h(\widetilde{\alpha}(u))\varphi] = h(\widetilde{\alpha}(u))D\varphi + \varphi\alpha(u)h'(\widetilde{\alpha}(u))Du$ almost everywhere in Ω and since the functions h and h' have a compact support, (2.6) and property (2.10) imply that the field $\mathbf{A}(x,u)Du \cdot D[h(\widetilde{\alpha}(u))\varphi]$ belongs to $L^1(\Omega)$. The sequence $h_n(\widetilde{\alpha}(u))$ converges to 1 almost everywhere in Ω as n goes to infinity and is bounded by 1, Lebesgue convergence theorem then allows us to conclude that

(2.56)

$$\begin{aligned} \lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega h_n(\widetilde{\alpha}(u^\varepsilon))\mathbf{A}^\varepsilon(x,u^\varepsilon)Du^\varepsilon \cdot D[h(\widetilde{\alpha}(u))\varphi]\,\mathrm{d}x &= \int_\Omega \mathbf{A}(x,u)Du \cdot D[h(\widetilde{\alpha}(u))\varphi]\,\mathrm{d}x \\ &= \int_\Omega h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du \cdot D\varphi\,\mathrm{d}x + \int_\Omega \alpha(u)h'(\widetilde{\alpha}(u))\varphi\mathbf{A}(x,u)Du \cdot Du\,\mathrm{d}x \end{aligned}$$

At last (2.18), (2.32) and the behavior of the sequence h_n together with Lebesgue convergence theorem lead to

$$\text{(2.57)} \qquad \lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega f^\varepsilon h_n(\widetilde{\alpha}(u^\varepsilon))h(\widetilde{\alpha}(u))\varphi\,\mathrm{d}x = \int_\Omega f h(\widetilde{\alpha}(u))\varphi\,\mathrm{d}x$$

Gathering (2.54), (2.55), (2.56) and (2.57) we conclude that

$$\int_\Omega h(\widetilde{\alpha}(u))\mathbf{A}(x,u)Du\cdot D\varphi\,\mathrm{d}x+\int_\Omega \alpha(u)h'(\widetilde{\alpha}(u))\varphi\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x=\int_\Omega fh(\widetilde{\alpha}(u))\varphi\,\mathrm{d}x,$$

that is (2.12).

The proof of Theorem 2.3 is now complete. □

2.4. **A few estimates and remarks on renormalized solutions.** This section is devoted to give essentially two estimates on any solution u of (1.1)–(1.2) (in the sense of Definition 2.1). These estimates are well known for renormalized solutions (see [14]) at least for non degenerate problems so that we will just sketch their proof.

Lemma 2.7. *Let u be a renormalized solution of* (1.1)–(1.2). *Then for any function $\varphi\in\mathcal{C}^1(\mathbb{R})$ such that $\varphi(0)=0$ we have*

$$\frac{1}{n}\int_{\{|\varphi(\widetilde{\alpha}(u))|\le n\}}\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)DuD\widetilde{\alpha}(u)\,\mathrm{d}x\longrightarrow 0\quad\text{as } n \text{ tends to } +\infty. \tag{2.58}$$

Moreover for any bounded and increasing function $\psi\in\mathcal{C}^1(\mathbb{R})$ such that $\psi(0)=0$, u satisfies

$$\mathbf{A}(x,u)DuD\psi\big(\widetilde{\alpha}(u)\big)\in L^1(\Omega). \tag{2.59}$$

Remark 2.8. Notice that (2.58) and (2.59) are generalizations of (2.10) and (2.11) respectively.

Sketch of the proof of Lemma 2.7. The program to be used to obtain estimates on a renormalized solution is classical : first one takes $h=h_p$ (where h_p is defined in (2.7) for any integer $p\ge 1$) in equation (2.12), then one multiplies (2.12) (with $h=h_p$) by an adequate admissible test function and one lets p tends to $+\infty$ by using (2.11). Remark that for any test function $z\in L^\infty(\Omega)\cap H^1_0(\Omega)$, the contribution of the second term in (2.12) in such a process is equal to 0 since

$$\Big|\int_\Omega h_p'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\widetilde{\alpha}(u)z\,\mathrm{d}x\Big|\le \frac{\|z\|_{L^\infty(\Omega)}}{p}\int_{\{|\widetilde{\alpha}(u)|\le p\}}\alpha(u)\mathbf{A}(x,u)Du\cdot Du\,\mathrm{d}x\longrightarrow 0\quad\text{as } p \text{ tends to } +\infty,$$

because of (2.11) and (2.13).

As a consequence one has for any function $z\in L^\infty(\Omega)\cap H^1_0(\Omega)$

$$\lim_{p\to+\infty}\int_\Omega h_p\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot Dz\,\mathrm{d}x=\int_\Omega f\,z\,\mathrm{d}x, \tag{2.60}$$

since $h_p\big(\widetilde{\alpha}(u)\big)\to 1$ almost everywhere in Ω and weakly-$*$ in $L^\infty(\Omega)$.

Indeed (2.60) is specially useful when the function z is such that one can identify the limit of the left hand side of (2.60) or at least a lower bound of this limit. This is the case if one first chooses $z=T_n\big(\varphi(\widetilde{\alpha}(u))\big)$ (which belongs to $L^\infty(\Omega)\cap H^1_0(\Omega)$ because of the properties of φ and of (2.9)) since for a fixed integer $n\ge 1$

$$\int_\Omega h_p\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot DT_n\big(\varphi(\widetilde{\alpha}(u))\big)\,\mathrm{d}x=\int_{\{|\varphi(\widetilde{\alpha}(u))|\le n\}}\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\widetilde{\alpha}(u)\,\mathrm{d}x$$

as soon as $p > n$. With this choice (2.60) gives for any $n \geq 1$

$$\int_{\{|\varphi(\widetilde{\alpha}(u))|\leq n\}} \varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\widetilde{\alpha}(u)\,\mathrm{d}x = \int_\Omega f\,T_n\big(\varphi(\widetilde{\alpha}(u))\big)\,\mathrm{d}x.$$

Multiplying the above equality by $1/n$ and letting n tends to $+\infty$ leads to (2.58) since $\frac{T_n(\varphi(\widetilde{\alpha}(u)))}{n} \to 0$ almost everywhere in Ω and weakly-$*$ in $L^\infty(\Omega)$ (recall that u is finite almost everywhere in Ω and that $\int_0^{+\infty}\alpha(s)\,\mathrm{d}s = \int_{-\infty}^0 \alpha(s)\,\mathrm{d}s = +\infty$).

To obtain (2.59) one chooses $z = \psi\big(T_n(\widetilde{\alpha}(u))\big)$ (which belongs to $L^\infty(\Omega)\cap H_0^1(\Omega)$ due to the properties of ψ and to (2.9)) and one remarks that

$$h_p\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\psi\big(T_n(\widetilde{\alpha}(u))\big) = \psi'\big(T_n(\widetilde{\alpha}(u))\big)\mathbf{A}(x,u)Du\cdot DT_n\big(\widetilde{\alpha}(u)\big)$$

almost everywhere in Ω, as soon as $p > n$. It follows that (2.60) implies that

$$\begin{aligned}\int_\Omega \psi'\big(T_n(\widetilde{\alpha}(u))\big)\mathbf{A}(x,u)Du\cdot DT_n\big(\widetilde{\alpha}(u)\big)\,\mathrm{d}x &= \int_\Omega f\,\psi\big(T_n(\widetilde{\alpha}(u))\big)\,\mathrm{d}x\\ &\leq \|f\|_{L^1(\Omega)}\,\|\psi\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

for any integer $n \geq 1$. Now $\psi'\big(T_n(\widetilde{\alpha}(u))\big)\mathbf{A}(x,u)Du\cdot DT_n\big(\widetilde{\alpha}(u)\big)$ is a positive function that converges almost everywhere in Ω to $\mathbf{A}(x,u)Du\cdot D\psi\big(\widetilde{\alpha}(u)\big)$ as n tends to $+\infty$ and Fatou's Lemma allows us to conclude that (2.59) holds true. □

3. A uniqueness result

It is well known that, as far as degenerate problems are involved and without any 0-order term, uniqueness results may not be expected on the solution u itself but on the field $\widetilde{\alpha}(u)$ (see the comments and the references in the introduction). In this section we address this uniqueness problem in the framework of solutions in the sense of Definition 2.1. The section is organized as follows. In Subsection 3.1 we detail the assumptions on the matrix $\mathbf{A}(x,s)$ that we use to obtain uniqueness of $\widetilde{\alpha}(u)$ and we give a few comments. In Subsection 3.2, we state Theorem 3.2 (uniqueness of $\widetilde{\alpha}(u)$). Subsection 3.3 is devoted to give the proof of Theorem 3.2.

3.1. **Assumptions and comments.** For the sake of simplicity, we assume in the whole section that the matrix $\mathbf{A}(x,s)$ is symmetric for almost any x in Ω and any s in $\mathbb{R}$. This is not essential to what follows but it avoids to state additional assumption on the behavior of the antisymmetric part of $\mathbf{A}(x,s)$ with respect to s.

We give below a sufficient condition on the dependence of the matrix field $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ with respect to s on the set $\{s\in\mathbb{R}\,;\,\alpha(s)\neq 0\}$ in order to ensure that the field $\widetilde{\alpha}(u)$ is uniquely determined for any solution given by Theorem 2.3. Loosely speaking we assume that there exists a smooth non decreasing function $\varphi(s)$ such that $\frac{\mathbf{A}(x,s)}{\alpha(s)\varphi'(\widetilde{\alpha}(s))}$ is a continuous matrix with respect to $\varphi(\widetilde{\alpha}(s))$ with a modulus of continuity controlled by a quantity that depends on $\varphi'(\widetilde{\alpha}(s))$ and $\varphi(\widetilde{\alpha}(s))$ (see condition (3.2) below).

We assume that

$$\mathbf{A}(x,s) \text{ is a symmetric matrix,} \tag{3.1}$$

$$\begin{cases} \text{there exist three constants } K_0,\ C_1 \text{ and } \delta > 1/2 \text{ and a real valued function } \varphi \in \mathcal{C}^1(\mathbb{R}) \text{ such that } \varphi'(s) \geq 1,\ \forall s \in \mathbb{R} \text{ and such that for any } s,\ t \in \mathbb{R} \\ \text{with } \alpha(s) \neq 0 \text{ and } \alpha(t) \neq 0 \text{ and for any } 0 \leq K \leq K_0 \text{ the matrix } \mathbf{A}(x,s) \text{ satisfies} \\ \left| \dfrac{\mathbf{A}(x,s)}{\alpha(s)\varphi'\big(\widetilde{\alpha}(s)\big)} - \dfrac{\mathbf{A}(x,t)}{\alpha(t)\varphi'\big(\widetilde{\alpha}(t)\big)} \right| \leq \dfrac{C_1 K}{\varphi'\big(\widetilde{\alpha}(s)\big)^{1/2}\varphi'\big(\widetilde{\alpha}(t)\big)^{1/2}\big(1+|\varphi(\widetilde{\alpha}(s))|+|\varphi(\widetilde{\alpha}(t))|\big)^{\delta}} \\ \text{almost everywhere in } \Omega, \text{ as soon as } \big|\varphi(\widetilde{\alpha}(s)) - \varphi(\widetilde{\alpha}(t))\big| < K. \end{cases} \tag{3.2}$$

Remark 3.1. Condition (3.2) implies that for any real number $M_0 \geq 0$, there exists a real number $M_1 \geq 0$ such that

$$\forall s \in \mathbb{R} \text{ such that } \alpha(s) \neq 0 \text{ and } \big|\widetilde{\alpha}(s)\big| \leq M_0, \text{ we have} \tag{3.3}$$
$$\left|\frac{\mathbf{A}(x,s)}{\alpha(s)\varphi'\big(\widetilde{\alpha}(s)\big)}\right| \leq M_1 \quad \text{almost everywhere in } \Omega.$$

Let us give a few comments on assumption (3.2).

In the case where there exists $\alpha_0 > 0$ such that $\mathbf{A}(x,s)\xi\cdot\xi \geq \alpha_0|\xi|^2$, for any $\xi \in \mathbb{R}^N$ and almost everywhere in Ω, one can choose $\alpha(s) = \alpha_0\ \forall s \in \mathbb{R}$ (even if this choice may be not the optimal one as far as (3.2) is concerned!). Then condition (3.2) reads as

$$\left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,t)}{\varphi'(t)}\right| \leq \frac{C_1 K \alpha_0}{\varphi'(s)^{1/2}\varphi'(t)^{1/2}\big(1+|\varphi(s)|+|\varphi(t)|\big)^{\delta}} \tag{3.4}$$

almost everywhere in Ω as soon as $|\varphi(t) - \varphi(s)| \leq K$.

Remark that since $\varphi'(s) \geq 1$, the function φ is indeed invertible and that φ^{-1} is continuous on $\mathbb{R}$. Then the function $\frac{\mathbf{A}(x,s)}{\varphi'(s)}$ may be expressed as a continuous function of $\varphi(s)$: $\frac{\mathbf{A}(x,s)}{\varphi(s)} = C(x,\varphi(s))$ for any $s \in \mathbb{R}$ and almost everywhere in Ω. In particular, condition (3.4) is satisfied if the modulus of continuity of $C(x,t)$, with respect to t, on any compact $[-M,M]$ of $\mathbb{R}$ is bounded by $\frac{C_1|t-s|}{\max_{s\in[-M,M]}\left[\varphi'(\varphi^{-1}(s))(1+2M)^{\delta}\right]}$ for almost everywhere $x \in \Omega$.

Still in the nondegenerate case, a more explicit condition on the dependence of $\mathbf{A}(x,s)$ with respect to s may be the following one. Assume that there exists a positive $\mathcal{C}^1$–function w defined on $\mathbb{R}$ with value in $\mathbb{R}$ such that

$$\big|\mathbf{A}(x,s) - \mathbf{A}(x,t)\big| \leq \left|\int_s^t w(z)\,\mathrm{d}z\right| \tag{3.5}$$

for any s and t in $\mathbb{R}$ and almost everywhere in Ω. Assume that the function w satisfies the following differential inequality

$$|w'| < C_2 w^{1+\eta} \quad \text{with } \eta > 0 \text{ and } C_2 > 0,$$

then (3.2) holds true. We postpone the proof of this result to the appendix (see Lemma A.1) at the end of the paper. As an example, let b be a non negative function belonging to $L^\infty(\Omega)$ such that $\operatorname{meas}\{x \in \Omega\,;\, b(x) = 0\} \neq 0$ and $\operatorname{meas}\{x \in \Omega\,;\, b(x) \neq 0\} \neq 0$ and let $\mathbf{B}$ be a coercive and symmetric matrix lying in $(L^\infty(\Omega))^{N\times N}$. Let us define the matrix $\mathbf{A}$ by

$$\mathbf{A}(x,s) = \big(1 + b(x)\exp(s)\sin^2(\exp(s^2))\big)\mathbf{B}(x) \quad \forall s \in \mathbb{R} \text{ and almost everywhere in } \Omega. \tag{3.6}$$

The conditions of Theorem 2.3 are satisfied with $\alpha(s) = 1$ (due to hypothesis meas$\{x \in \Omega\,;\, b(x) = 0\} \neq 0$) and (3.5) is also satisfied with $w(s) = M \exp(s + s^2)(s^2 + 1)$ where M depends on $\|b\|_{L^\infty(\Omega)}$.

The above example shows that, in the non degenerate case, assumption (3.2) allows us to consider a large class of operator with e.g. fast growth and fast oscillations at infinity.

In the case where the matrix $\mathbf{A}(x, s)$ degenerates when $\alpha(s) = 0$, condition (3.2) may be formally derived from the non degenerate issue through the following argument. Equation (1.1) is formally written as $-\operatorname{div}\left[\frac{\mathbf{A}(x,u)}{\alpha(u)} D\widetilde{\alpha}(u)\right] = f$ in Ω where the matrix $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ does not degenerate since $\frac{\mathbf{A}(x,s)}{\alpha(s)}\xi \cdot \xi \geq |\xi|^2$ by (2.3). Then assumption (3.2) is obtained through considering the above equation together with $\widetilde{\alpha}(u) = 0$ on $\partial\Omega$ as a coercive problem for the unknown $\widetilde{\alpha}(u)$ and therefore by replacing $\mathbf{A}(x, s)$ by $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ and $\varphi(s)$, $\varphi'(s)$ by $\varphi\big(\widetilde{\alpha}(s)\big)$, $\varphi'\big(\widetilde{\alpha}(s)\big)$ in our condition (3.4) for non degenerate problems. Although assumption (3.4) authorizes both fast growth and fast oscillations of the matrix $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ with respect to $\widetilde{\alpha}(s)$, it is a strong assumption since it implies that $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ is locally uniformly continuous with respect to $\widetilde{\alpha}(s)$. As an example condition (3.2) is not satisfied in the apparently simple problem (1.1)–(1.2) with

$$\mathbf{A}(x, s) = \lambda(x)(s - 1)^+ + \mu(x)s^- \tag{3.7}$$

where $\lambda(x)$, $\mu(x)$ are $L^\infty(\Omega)$–function such that $\lambda(x) \geq 1$, $\mu(x) \geq 1$ almost everywhere in Ω. Indeed in this example, one has $\alpha(s) = (s - 1)^+ + s^-$ so that $\frac{\mathbf{A}(x,s)}{\alpha(s)} = \lambda(x)$ for $\widetilde{\alpha}(s) > 0$ and $\frac{\mathbf{A}(x,s)}{\alpha(s)} = \mu(x)$ for $\widetilde{\alpha}(s) < 0$. If $\lambda(x) \neq \mu(x)$ in a subset of Ω it follows that $\frac{\mathbf{A}(x,s)}{\alpha(s)}$ is not continuous at $\widetilde{\alpha}(s) = 0$ for such x's and (3.2) is not satisfied. Remark that Theorem 2.3 allows to obtain a solution of (1.1)–(1.2) with $\mathbf{A}(x, s)$ given by (3.7), but that we cannot apply Theorem 3.2 to claim that this solution is unique.

3.2. **Uniqueness result.** Our uniqueness result is the following theorem.

Theorem 3.2. *Assume that the assumptions of Theorem 2.3 hold true and moreover that* $\mathbf{A}(x, s)$ *satisfies the assumptions* (3.1)–(3.2). *Then, the field* $\widetilde{\alpha}(u)$ *(whose existence is ensured by Theorem 2.3) is unique.*

Remark 3.3. Indeed Theorem 3.2 implies that if u and v are two solutions in the sense of Definition 2.1, then

$$\Omega_1 = \{x \in \Omega\,;\, \alpha(u)(x) = 0\} = \{x \in \Omega\,;\, \alpha(v)(x) = 0\}$$

and

$$u = v \quad \text{almost everywhere on } \Omega \setminus \Omega_1.$$

In the non degenerate case we obtain in particular.

Corollary 3.4. *Assume that there exists a positive constant* $\alpha_0 > 0$ *such that* $\alpha(s) \geq \alpha_0$ $\forall s \in \mathbb{R}$. *Under the assumptions of Theorem 3.2, the solution* u *of* (1.1)–(1.2) *(in the sense of Definition 2.1) is unique.*

As mentioned in the introduction, similar results to the one stated in Corollary 3.4 have been recently obtained by A. Porretta (see [19]) for entropy solutions of Problem of type (1.1)–(1.2).

3.3. **Proof of Theorem 3.2.**

Let u and v be two renormalized solutions of (1.1)–(1.2) in the sense of Definition 2.1. Let $\varphi \in \mathcal{C}^1(\mathbb{R})$ be a function such that (3.2) holds true. Remark first that one can choose $h(t) = h_n\big(\varphi(t)\big)$, $\forall t \in \mathbb{R}$ (where h_n is defined in (2.7)) in (2.12) for u and v, since indeed $h_n\big(\varphi(t)\big) \in W^{1,\infty}(\mathbb{R})$ with compact support (due to $\varphi'(t) \geq 1$, $\forall t \in \mathbb{R}$).

Equation (2.12) then gives for any $n \geq 1$

$$(3.8)\quad -\operatorname{div}\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big)\mathbf{A}(x,u)Du\big] + h_n'\big(\varphi(\widetilde{\alpha}(u))\big)\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du \cdot D\widetilde{\alpha}(u) = f h_n\big(\varphi(\widetilde{\alpha}(u))\big) \quad \text{in } \mathcal{D}'(\Omega),$$

the same equation being true for v.

In order to prove that $\widetilde{\alpha}(u) = \widetilde{\alpha}(v)$, the usual technique consists to take the difference of the two equations (3.8) for u and v and to plug an adequate test function in the resulting equation. According to the meaning of (3.8), such a test function must belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Remark that for arbitrary real numbers $p \geq 0$ and $q \geq 0$, both $T_p\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)$ and $T_q\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$ because of (2.9) and $\varphi'(t) \geq 1$, $\forall t \in \mathbb{R}$. For fixed real numbers $K \geq 0$, $p \geq 0$, $q \geq 0$, we the use $T_K\big[T_p\big(\varphi\big(\widetilde{\alpha}(u)\big)\big) - T_q\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\big]$ as a test function in $(3.8)_u$–$(3.8)_v$. It is then easy to pass to the limit as p and q (successively) tend to $+\infty$ for fixed $K \geq 0$ and $n \geq 1$. Indeed, denoting by $M_n \geq 0$ a real number such that $\operatorname{supp}(h_n \circ \varphi) \subset [-M_n, M_n]$, one has firstly

$$h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)DT_K\big[T_p\big(\varphi\big(\widetilde{\alpha}(u)\big)\big) - T_q\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\big] = h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)DT_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]$$

as soon as $p > M_n$ and $q > M_n + K$. Secondly

$$T_K\big[T_p\big(\varphi\big(\widetilde{\alpha}(u)\big)\big) - T_q\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\big] \longrightarrow T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]$$

almost everywhere in Ω and weakly–$*$ in $L^\infty(\Omega)$ as p and q tend (successively) to $+\infty$.

Through such a process, we obtain the following equation, which could be formally derived by plugging the test function $T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]$ in $(3.8)_u$–$(3.8)_v$

$$\begin{aligned}(3.9)\quad &\int_\Omega \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big)\mathbf{A}(x,u)Du - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\mathbf{A}(x,v)Dv\big] \cdot DT_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\,\mathrm{d}x \\ &+ \int_\Omega h_n'\big(\varphi(\widetilde{\alpha}(u))\big)\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du \cdot D\widetilde{\alpha}(u)T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\,\mathrm{d}x \\ &- \int_\Omega h_n'\big(\varphi(\widetilde{\alpha}(v))\big)\varphi'\big(\widetilde{\alpha}(v)\big)\mathbf{A}(x,v)Dv \cdot D\widetilde{\alpha}(v)T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\,\mathrm{d}x \\ &= \int_\Omega f\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\,\mathrm{d}x,\end{aligned}$$

for any $K \geq 0$ and any $n \geq 1$.

We propose to pass to the limit (or the limit–sup) in (3.9) as n tends to $+\infty$ first and then as K tends to 0. An attentive reader should notice that the reverse is usually performed in the uniqueness proof program when dealing with renormalized solutions. But this last order in passing to the limit in the parameters (i.e. $K \to 0$ and then $n \to +\infty$) leads to uniqueness of the solution only in the case where a term of order 0, as an example of the form λu with $\lambda > 0$, is added to the operator in (1.1). In this case only local assumptions on $\mathbf{A}(x,s)$ with respect to s are sufficient, because it is enough to show that the contribution of the first term in (3.9) has a non negative limit as $K \to 0$ and then $n \to +\infty$. As far as (1.1) is concerned, namely without any term of order 0, the uniqueness character of $\widetilde{\alpha}(u)$ must be recovered from the first term of (3.9) letting first $n \to +\infty$ and then $K \to 0$. This is the reason why we need a global assumption of the type (3.9) on $\mathbf{A}(x,s)$.

Let us turn back to (3.9) and to shorten the notations let us introduce the subsets of Ω

$$\Omega_0^u = \{x \in \Omega\,;\, \alpha(u)(x) \neq 0\}, \quad \Omega_0^v = \{x \in \Omega\,;\, \alpha(v)(x) \neq 0\}, \tag{3.10}$$

$$\Omega_1^u = \{x \in \Omega\,;\, \alpha(u)(x) = 0\}, \quad \Omega_1^v = \{x \in \Omega\,;\, \alpha(v)(x) = 0\}, \tag{3.11}$$

and the positive quantities

$$\omega_n^u = \Big| \int_\Omega h_n'\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du \cdot D\widetilde{\alpha}(u)\,\mathrm{d}x\Big| \tag{3.12}$$

$$\omega_n^v = \Big| \int_\Omega h_n'\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\varphi'\big(\widetilde{\alpha}(v)\big)\mathbf{A}(x,v)Dv \cdot D\widetilde{\alpha}(v)\,\mathrm{d}x\Big| \tag{3.13}$$

and, for K fixed, the function

$$W_K = T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]. \tag{3.14}$$

Then in (3.9), the first integral over Ω is split into two integrals over $\Omega_0^u \cap \Omega_0^v$ and over $\Omega_1^u \cup \Omega_1^v$. Noticing that, in view of (2.4), $\mathbf{A}(x,u) = 0$ almost everywhere on Ω_1^u and $\mathbf{A}(x,v) = 0$ almost everywhere on Ω_1^v, we obtain using (2.13)

$$\begin{aligned}
&\int_{\Omega_0^u \cap \Omega_0^v} \Big[h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u) - h_n\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\frac{\mathbf{A}(x,v)}{\alpha(v)}D\widetilde{\alpha}(v)\Big] \cdot DW_K\,\mathrm{d}x \\
&\qquad + \int_{\Omega_0^u \cap \Omega_1^v} h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u) \cdot DW_K\,\mathrm{d}x \\
&\qquad - \int_{\Omega_0^v \cap \Omega_1^u} h_n\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\frac{\mathbf{A}(x,v)}{\alpha(v)}D\widetilde{\alpha}(v) \cdot DW_K\,\mathrm{d}x \\
&\qquad\qquad \leq K\omega_n^u + K\omega_n^v + \int_\Omega f\big[h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big) - h_n\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\big]W_K\,\mathrm{d}x,
\end{aligned} \tag{3.15}$$

for any $n \geq 1$ and any $K \geq 0$.

The first term in (3.15) is derived as follows

$$\begin{aligned}
&\int_{\Omega_0^u \cap \Omega_0^v} \Big[h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big)\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u) - h_n\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\frac{\mathbf{A}(x,v)}{\alpha(v)}D\widetilde{\alpha}(v)\Big] \cdot DW_K\,\mathrm{d}x \\
=&\frac{1}{2}\int_{\Omega_0^u \cap \Omega_0^v} \big[h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big) + h_n\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\big]\Big[\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u) - \frac{\mathbf{A}(x,v)}{\alpha(v)}D\widetilde{\alpha}(v)\Big] \cdot DW_K\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{\Omega_0^u \cap \Omega_0^v} \big[h_n\big(\varphi\big(\widetilde{\alpha}(u)\big)\big) - h_n\big(\varphi\big(\widetilde{\alpha}(v)\big)\big)\big]\Big[\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u) + \frac{\mathbf{A}(x,v)}{\alpha(v)}D\widetilde{\alpha}(v)\Big] \cdot DW_K\,\mathrm{d}x \\
=&A_n^K + B_n^K,
\end{aligned} \tag{3.16}$$

for any $n \geq 1$ and any $K \geq 0$. Remark that for fixed $n \geq 1$ and $K \geq 0$, the quantities A_n^K and B_n^K are well defined because in all the above integrals (recalling that $W_K = T_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]$) both the fields $\widetilde{\alpha}(u)$ and $\widetilde{\alpha}(v)$ are truncated since $\mathrm{supp}(T_K')$ and $\mathrm{supp}(h_n \circ \varphi)$ are compact. Then we have

$$A_n^K = \frac{1}{2}\int_{\Omega_0^u\cap\Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \times \Big[\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))}D\varphi\big(\widetilde{\alpha}(u)\big) - \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}D\varphi\big(\widetilde{\alpha}(v)\big)\Big]\cdot DW_K\,\mathrm{d}x$$

which is licit, since $\varphi'(t) \geq 1$, $\forall t \in \mathbb{R}$, and implies

$$\begin{aligned}
A_n^K =& \frac{1}{4}\int_{\Omega_0^u\cap\Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \\
&\times \Big[\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))} - \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}\Big] D\big[\varphi\big(\widetilde{\alpha}(u)\big) + \varphi\big(\widetilde{\alpha}(v)\big)\big]\cdot DW_K\,\mathrm{d}x \\
&+ \frac{1}{4}\int_{\Omega_0^u\cap\Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \\
&\times \Big[\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))} + \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}\Big] D\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\cdot DW_K\,\mathrm{d}x \\
=& C_n^K + D_n^K,
\end{aligned} \tag{3.17}$$

for any $n \geq 1$ and any $K \geq 0$. Remark that the quantities C_n^K and D_n^K are well defined because in all the above integrals both $\widetilde{\alpha}(u)$ and $\widetilde{\alpha}(v)$ are truncated and because (3.3) then implies that $\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))}$ and $\frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}$ are bounded on the subsets where $\widetilde{\alpha}(u)$ and $\widetilde{\alpha}(v)$ are bounded.

Collecting together (3.15), (3.16), (3.17) leads to

$$\begin{aligned}
&\frac{1}{4}\int_{\Omega_0^u\cap\Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \\
&\qquad\times \Big[\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))} + \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}\Big] D\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\cdot DW_K\,\mathrm{d}x \\
&\qquad+ \int_{\Omega_0^u\cap\Omega_1^v} h_n\big(\varphi(\widetilde{\alpha}(u))\big)\frac{\mathbf{A}(x,u)}{\alpha(u)}D\widetilde{\alpha}(u)\cdot DW_K\,\mathrm{d}x \\
&\qquad- \int_{\Omega_0^v\cap\Omega_1^u} h_n\big(\varphi(\widetilde{\alpha}(v))\big)\frac{\mathbf{A}(x,v)}{\alpha(v)}D\widetilde{\alpha}(v)\cdot DW_K\,\mathrm{d}x \\
&\leq K\omega_n^u + K\omega_n^v + \int_\Omega f\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]W_K\,\mathrm{d}x + |B_n^K| + |C_n^K|,
\end{aligned} \tag{3.18}$$

for any $n \geq 1$ and any $K \geq 0$.

As far as the term $|B_n^K|$ is concerned, we use the symmetric character of the matrices $\mathbf{A}(x,u)$ and $\mathbf{A}(x,v)$ to obtain for any $\varepsilon > 0$

$$
\begin{aligned}
(3.19)\quad |B_n^K| \leq & \frac{\varepsilon}{2}\int_{\Omega_0^u\cap\Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \\
&\qquad \times\Big[\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))} + \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}\Big] DW_K\cdot DW_K \,\mathrm{d}x \\
&+ \frac{1}{2\varepsilon}\int_{\Omega_0^u\cap\Omega_0^v\cap\{|\varphi(\widetilde{\alpha}(u))-\varphi(\widetilde{\alpha}(v))|\leq K\}} \big|h_n\big(\varphi(\widetilde{\alpha}(u))\big) - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big| \\
&\times\Big[\varphi'\big(\widetilde{\alpha}(u)\big)\frac{\mathbf{A}(x,u)}{\alpha(u)} D\widetilde{\alpha}(u)\cdot D\widetilde{\alpha}(u) + \varphi'\big(\widetilde{\alpha}(v)\big)\frac{\mathbf{A}(x,v)}{\alpha(v)} D\widetilde{\alpha}(v)\cdot D\widetilde{\alpha}(v)\Big]\,\mathrm{d}x,
\end{aligned}
$$

where we have used again $T_K'(s) = \mathbb{1}_{\{|s|<K\}}$ for $|s| \neq K$.

As far as the term C_n^K is concerned, we use assumption (3.2) which implies that for any $0 \leq K < K_0$

$$
\begin{aligned}
(3.20)\quad & \Big|\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'\big(\widetilde{\alpha}(u)\big)} - \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'\big(\widetilde{\alpha}(v)\big)}\Big| \\
&\qquad \leq \frac{C_1 K}{\varphi'\big(\widetilde{\alpha}(u)\big)^{1/2}\varphi'\big(\widetilde{\alpha}(v)\big)^{1/2}\big(1+|\varphi(\widetilde{\alpha}(u))|+|\varphi(\widetilde{\alpha}(v))|\big)^{\delta}}
\end{aligned}
$$

almost everywhere in $\Omega_0^u\cap\Omega_0^v\cap\{|\varphi(\widetilde{\alpha}(u))-\varphi(\widetilde{\alpha}(v))| \leq K\}$.

With the help of Young's inequality, the definition of C_n^K (in (3.17)) and (3.20) give for any $\varepsilon > 0$

$$
\begin{aligned}
(3.21)\quad |C_n^K| \leq & C_1\frac{K^2}{4\varepsilon}\int_{\substack{\Omega_0^u\cap\Omega_0^v\cap\{\varphi(\widetilde{\alpha}(u))\neq\varphi(\widetilde{\alpha}(v))\}\\ \cap\{|\varphi(\widetilde{\alpha}(u))-\varphi(\widetilde{\alpha}(v))|\leq K\}}} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \\
&\qquad\times\frac{\Big[\varphi'(\widetilde{\alpha}(v))\big|D\varphi\big(\widetilde{\alpha}(u)\big)\big|^2 + \varphi'(\widetilde{\alpha}(u))\big|D\varphi\big(\widetilde{\alpha}(v)\big)\big|^2\Big]}{\varphi'\big(\widetilde{\alpha}(u)\big)\varphi'\big(\widetilde{\alpha}(v)\big)\big(1+|\varphi(\widetilde{\alpha}(u))|+|\varphi(\widetilde{\alpha}(v))|\big)^{2\delta}}\,\mathrm{d}x \\
&+ \frac{\varepsilon}{4}\int_{\Omega_0^u\cap\Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]\Big[\frac{1}{\varphi'(\widetilde{\alpha}(u))} + \frac{1}{\varphi'(\widetilde{\alpha}(v))}\Big]|DW_K|^2\,\mathrm{d}x.
\end{aligned}
$$

Gathering (3.18), (3.19), (3.21) and using $\frac{\mathbf{A}(x,s)}{\alpha(s)} \geq I$ for any s such that $\alpha(s) \neq 0$, we obtain choosing ε small enough

$$
\begin{aligned}
(3.22)\quad & \int_{\Omega_0^u \cap \Omega_0^v} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]\Big[\frac{\mathbf{A}(x,u)}{\alpha(u)\varphi'(\widetilde{\alpha}(u))} + \frac{\mathbf{A}(x,v)}{\alpha(v)\varphi'(\widetilde{\alpha}(v))}\Big] DW_K \cdot DW_K \,\mathrm{d}x \\
& + \int_{\Omega_0^u \cap \Omega_1^v} h_n\big(\varphi(\widetilde{\alpha}(u))\big)\frac{\mathbf{A}(x,u)}{\alpha(u)} D\widetilde{\alpha}(u) \cdot DW_K \,\mathrm{d}x \\
& - \int_{\Omega_0^v \cap \Omega_1^u} h_n\big(\varphi(\widetilde{\alpha}(v))\big)\frac{\mathbf{A}(x,v)}{\alpha(v)} D\widetilde{\alpha}(v) \cdot DW_K \,\mathrm{d}x \\
& \leq C\Big\{ K\omega_n^u + K\omega_n^v + \int_\Omega f\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] W_K \,\mathrm{d}x \\
& + \int_{\Omega_0^u \cap \Omega_0^v \cap \{|\varphi(\widetilde{\alpha}(u)) - \varphi(\widetilde{\alpha}(v))| \leq K\}} \big|h_n\big(\varphi(\widetilde{\alpha}(u))\big) - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big| \\
& \quad \times \Big[\varphi'\big(\widetilde{\alpha}(u)\big)\frac{\mathbf{A}(x,u)}{\alpha(u)} D\widetilde{\alpha}(u) \cdot D\widetilde{\alpha}(u) + \varphi'\big(\widetilde{\alpha}(v)\big)\frac{\mathbf{A}(x,v)}{\alpha(v)} D\widetilde{\alpha}(v) \cdot D\widetilde{\alpha}(v)\Big] \,\mathrm{d}x \\
& + K^2 \int_{\substack{\Omega_0^u \cap \Omega_0^v \cap \{\varphi(\widetilde{\alpha}(u)) \neq \varphi(\widetilde{\alpha}(v))\} \\ \cap \{|\varphi(\widetilde{\alpha}(u)) - \varphi(\widetilde{\alpha}(v))| \leq K\}}} \big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) + h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big] \\
& \quad \times \frac{\Big[\varphi'(\widetilde{\alpha}(v))\big|D\varphi\big(\widetilde{\alpha}(u)\big)\big|^2 + \varphi'(\widetilde{\alpha}(u))\big|D\varphi\big(\widetilde{\alpha}(v)\big)\big|^2\Big]}{\varphi'\big(\widetilde{\alpha}(u)\big)\varphi'\big(\widetilde{\alpha}(v)\big)\big(1 + |\varphi(\widetilde{\alpha}(u))| + |\varphi(\widetilde{\alpha}(v))|\big)^{2\delta}} \,\mathrm{d}x \Big\}
\end{aligned}
$$

for any $n \geq 1$ and any $0 \leq K < K_0$ and where C is a generic constant independent of n and K.

In (3.22) we first use the inequalities $h_n(s)h_n(t) \leq \inf\big(h_n(t), h_n(s)\big)$, $\forall t \in \mathbb{R}$, $\forall s \in \mathbb{R}$ and (again) $\frac{\mathbf{A}(x,s)}{\alpha(s)} \geq I$ for any $s \in \mathbb{R}$ such that $\alpha(s) \neq 0$. Secondly we remark that

$$
\begin{aligned}
D\widetilde{\alpha}(u) \cdot DW_K &= D\widetilde{\alpha}(u) \cdot DT_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big] \\
&= \frac{1}{\varphi'(\widetilde{\alpha}(u))}\big[D\varphi\big(\widetilde{\alpha}(u)\big) - D\varphi\big(\widetilde{\alpha}(v)\big)\big] \cdot DT_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big] \\
&= \frac{1}{\varphi'(\widetilde{\alpha}(u))}\big|DT_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\big|^2 \quad \text{almost everywhere in } \Omega_1^v,
\end{aligned}
$$

and

$$
-D\widetilde{\alpha}(v) \cdot DW_K = \frac{1}{\varphi'(\widetilde{\alpha}(v))}\big|DT_K\big[\varphi\big(\widetilde{\alpha}(u)\big) - \varphi\big(\widetilde{\alpha}(v)\big)\big]\big|^2 \quad \text{almost everywhere in } \Omega_1^u,
$$

because of (2.19) (which holds true for u and v).

Thirdly, we recall that h_n is a Lipschitz continuous function with $\|h_n'\|_{L^\infty(\mathbb{R})} \le \frac{1}{n}$ and that $\varphi'(t) \ge 1$, $\forall t \in \mathbb{R}$. As a consequence of all these facts, (3.22) implies that

$$\begin{aligned}(3.23)\quad &\int_\Omega h_n\big(\varphi(\widetilde{\alpha}(u))\big)h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big|DT_K\big[\varphi(\widetilde{\alpha}(u))-\varphi(\widetilde{\alpha}(v))\big]\big|^2\Big(\frac{1}{\varphi'(\widetilde{\alpha}(u))\varphi'(\widetilde{\alpha}(v))}\Big)\,\mathrm{d}x\\ &\le C\Bigg\{K\omega_n^u + K\omega_n^v + \int_\Omega f\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big) - h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]W_K\,\mathrm{d}x\\ &+\frac{K}{n}\int_{\substack{\Omega\cap\{|\varphi(\widetilde{\alpha}(u))|\le 2n+K\}\\ \cap\{|\varphi(\widetilde{\alpha}(v))|\le 2n+K\}}}\Big[\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\widetilde{\alpha}(u)+\varphi'\big(\widetilde{\alpha}(v)\big)\mathbf{A}(x,v)Dv\cdot D\widetilde{\alpha}(v)\Big]\,\mathrm{d}x\\ &+K^2\int_{\substack{\Omega_0^u\cap\Omega_0^v\cap\{\varphi(\widetilde{\alpha}(u))\ne\varphi(\widetilde{\alpha}(v))\}\\ \cap\{|\varphi(\widetilde{\alpha}(u))-\varphi(\widetilde{\alpha}(v))|\le K\}}}\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big)+h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]\\ &\times\Big[\frac{\big|D\varphi(\widetilde{\alpha}(u))\big|^2}{\varphi'\big(\widetilde{\alpha}(u)\big)\big[1+|\varphi(\widetilde{\alpha}(u))|\big]^{2\delta}}+\frac{\big|D\varphi(\widetilde{\alpha}(v))\big|^2}{\varphi'\big(\widetilde{\alpha}(v)\big)\big[1+|\varphi(\widetilde{\alpha}(v))|\big]^{2\delta}}\Big]\,\mathrm{d}x\Bigg\}.\end{aligned}$$

Now we pass to the limit as n tends to $+\infty$ in (3.23) (for any fixed real number $K \ge 0$). Due to the definition of ω_n^u and ω_n^v in (3.12) and (3.13) and the definition of h_n in (2.7) we first obtain using estimate (2.58) of Lemma 2.7

$$(3.24)\qquad \lim_{n\to+\infty}\omega_n^u = \lim_{n\to+\infty}\omega_n^v = 0.$$

Since $h_n\big(\varphi(\widetilde{\alpha}(u))\big)$ and $h_n\big(\varphi(\widetilde{\alpha}(v))\big)$ both converge to 1 almost everywhere in Ω and in $L^\infty(\Omega)$ weak–$*$, we also have

$$\lim_{n\to+\infty}\int_\Omega f\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big)-h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]W_K\,\mathrm{d}x = 0.$$

Due to the properties of φ in assumption (3.2), estimate (2.58) of Lemma 2.7 shows that

$$(3.25)\qquad \lim_{n\to+\infty}\frac{K}{n}\int_{\substack{\Omega\cap\{|\varphi(\widetilde{\alpha}(u))|\le 2n+K\}\\ \cap\{|\varphi(\widetilde{\alpha}(v))|\le 2n+K\}}}\Big[\varphi'\big(\widetilde{\alpha}(u)\big)\mathbf{A}(x,u)Du\cdot D\widetilde{\alpha}(u)+\varphi'\big(\widetilde{\alpha}(v)\big)\mathbf{A}(x,v)Dv\cdot D\widetilde{\alpha}(v)\Big]\,\mathrm{d}x = 0.$$

At last, as far as the last term in (3.23) is concerned, let us define the function

$$\psi(t) = \int_0^{\varphi(t)}\frac{\mathrm{d}s}{(1+|s|)^{2\delta}}.$$

We have

$$\begin{aligned}&\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big)+h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]\Big[\frac{\big|D\varphi(\widetilde{\alpha}(u))\big|^2}{\varphi'\big(\widetilde{\alpha}(u)\big)\big[1+|\varphi(\widetilde{\alpha}(u))|\big]^{2\delta}}+\frac{\big|D\varphi(\widetilde{\alpha}(v))\big|^2}{\varphi'\big(\widetilde{\alpha}(v)\big)\big[1+|\varphi(\widetilde{\alpha}(v))|\big]^{2\delta}}\Big]\\ &\quad\le\big[h_n\big(\varphi(\widetilde{\alpha}(u))\big)+h_n\big(\varphi(\widetilde{\alpha}(v))\big)\big]\big[\mathbf{A}(x,u)Du\cdot D\psi(\widetilde{\alpha}(u))+\mathbf{A}(x,v)Dv\cdot D\psi(\widetilde{\alpha}(v))\big]\end{aligned}$$

almost everywhere on the subset $\{\big|\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big|<K\}$.

Since $\delta > \frac{1}{2}$, ψ is an admissible function to apply (2.59) of Lemma 2.7 which gives

$$(3.26)\qquad \mathbf{A}(x,u)Du\cdot D\psi(\widetilde{\alpha}(u)) \in L^1(\Omega).$$

As a consequence of (3.24), (3.25) and (3.26) and with the use of Fatou's lemma, we are in a position to pass to the limit as n tends to $+\infty$ in (3.23) to obtain

$$\int_\Omega \frac{1}{\varphi'\big(\widetilde{\alpha}(u)\big)\varphi'\big(\widetilde{\alpha}(v)\big)}\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|^2\,\mathrm{d}x \le CK^2\int_{\substack{\Omega_0^u\cap\Omega_0^v\cap\{\varphi(\widetilde{\alpha}(u))\neq\varphi(\widetilde{\alpha}(v))\}\\ \cap\{|\varphi(\widetilde{\alpha}(u))-\varphi(\widetilde{\alpha}(v))|\le K\}}}\big(\mathbf{A}(x,u)Du\cdot D\psi(u)+\mathbf{A}(x,v)Dv\cdot D\psi(v)\big)\,\mathrm{d}x. \tag{3.27}$$

Remark that the fact that the integrand in the first term of (3.27) belongs to $L^1(\Omega)$ is not a direct consequence of Definition 2.1 since φ is not assumed to have a compact support.

Now we let K tends to 0 in (3.27) divided by K^2 which gives, using again $\mathbf{A}(x,u)Du\cdot D\psi(u)\in L^1(\Omega)$, $\mathbf{A}(x,v)Dv\cdot D\psi(v)\in L^1(\Omega)$ and Lebesgue's Theorem,

$$\lim_{K\to 0}\frac{1}{K^2}\int_\Omega \frac{1}{\varphi'\big(\widetilde{\alpha}(u)\big)\varphi'\big(\widetilde{\alpha}(v)\big)}\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|^2\,\mathrm{d}x=0. \tag{3.28}$$

To conclude the proof of Theorem 3.2, we show below that (2.11) and (3.28) imply that $\widetilde{\alpha}(u)=\widetilde{\alpha}(v)$ almost everywhere in Ω. To this end, we first remark that, for any integer $n\ge 1$, $h_n\big(\widetilde{\alpha}(u)\big)T_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)$ belongs to $H_0^1(\Omega)$ because of (2.2) and (2.9). Poincaré's inequality then gives

$$\int_\Omega h_n\big(\widetilde{\alpha}(u)\big)\Big|\frac{T_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)}{K}\Big|\,\mathrm{d}x \le C\Big\{\int_\Omega h_n'\big(\widetilde{\alpha}(u)\big)\big|D\widetilde{\alpha}(u)\big|\,\mathrm{d}x+\int_\Omega\frac{h_n\big(\widetilde{\alpha}(u)\big)}{K}\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|\,\mathrm{d}x\Big\}, \tag{3.29}$$

for any $n\ge 1$ and any $K\ge 0$ and where C is a constant independent of n and K.

We first let K tends to 0 and then n tend to $+\infty$ in (3.29). We first have for any $n\ge 1$ and any $1>K>0$

$$\int_\Omega\frac{h_n\big(\widetilde{\alpha}(u)\big)}{K}\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|\,\mathrm{d}x \le \frac{1}{K}\int_{\{|\varphi(\widetilde{\alpha}(u))|\le 2n\}\cap\{|\varphi(\widetilde{\alpha}(v))|\le 2n+1\}\}}\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|\,\mathrm{d}x.$$

Denoting by C_n a positive real number such that (recall (2.2) and that $\varphi'(s)\ge 1$ $\forall s\in\mathbb{R}$)

$$\{s\in\mathbb{R}\,;\,\big|\varphi\big(\widetilde{\alpha}(s)\big)\big|\le 2n\}\subset\{t\in\mathbb{R}\,;\,\big|\varphi\big(\widetilde{\alpha}(t)\big)\big|\le 2n+1\}\subset[-C_n,C_n],$$

we obtain

$$\int_\Omega\frac{h_n\big(\widetilde{\alpha}(u)\big)}{K}\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|\,\mathrm{d}x \le \frac{1}{K}\max_{s\in[-C_n,C_n]}\big(\varphi'(s)\big)\int_\Omega\frac{\big|DT_K\big(\varphi\big(\widetilde{\alpha}(u)\big)-\varphi\big(\widetilde{\alpha}(v)\big)\big)\big|}{\big(\varphi'(\widetilde{\alpha}(u))\big)^{1/2}\big(\varphi'(\widetilde{\alpha}(v))\big)^{1/2}}\,\mathrm{d}x.$$

In view of (3.28) and taking the limit as K tends to 0 in (3.29), it follows that, for any $n\ge 1$

$$\int_{\{\varphi(\widetilde{\alpha}(u))\neq\varphi(\widetilde{\alpha}(v))\}}h_n\big(\widetilde{\alpha}(u)\big)\,\mathrm{d}x\le\frac{C}{n}\int_{\{|\widetilde{\alpha}(u)|\le 2n\}}\big|D\widetilde{\alpha}(u)\big|\,\mathrm{d}x. \tag{3.30}$$

Now (2.3), (2.11) and (2.13) imply that the right hand side of (3.30) tends to 0 as n tends to $+\infty$, so that we obtain $\varphi\big(\widetilde{\alpha}(u)\big) = \varphi\big(\widetilde{\alpha}(v)\big)$ almost everywhere in Ω (recalling that $h_n\big(\widetilde{\alpha}(u)\big)$ converges to 1 almost everywhere in Ω and in $L^\infty(\Omega)$ weak–$*$ as n tends to $+\infty$). Since again $\varphi'(s) \geq 1$ $\forall s \in \mathbb{R}$, we finally conclude that $\widetilde{\alpha}(u) = \widetilde{\alpha}(v)$ almost everywhere in Ω and the proof of Theorem 3.2 is complete.

Appendix A. Proof of Lemma A.1

Lemma A.1. *Assume that* $\mathbf{A}(x,s)$ *satisfies* (2.2)–(2.6) *with* $\alpha(s) = \alpha_0$, $\forall s \in \mathbb{R}$, ($\alpha_0 > 0$) *and assume that there exists* $w \in \mathcal{C}^1(\mathbb{R})$ *such that*

$$\big|\mathbf{A}(x,s) - \mathbf{A}(x,t)\big| \leq \left| \int_s^t w(z)\,\mathrm{d}z \right| \tag{A.1}$$

$$w \geq 0 \tag{A.2}$$

$$|w'| \leq C_2 w^{1+\eta}, \tag{A.3}$$

where $C_2 > 0$ *and* $\eta > 0$. *Then the matrix* $\mathbf{A}(x,s)$ *verifies condition* (3.2).

Sketch of proof. The proof relies on the construction of the function φ using standard real analysis. We leave some details to the reader. Since $\alpha(s) = \alpha_0$ we can assume that $\alpha_0 = 1$ and then $\widetilde{\alpha}(t) = t$, $\forall t \in \mathbb{R}$.
Step 1. For any $n \geq 2$, let us define the function g_n by

$$\forall t \geq 0, \quad g_n(t) = \left[\int_0^t \big(1 + |w'(z)| + |w'(-z)|\big)\,\mathrm{d}z + w(0) + 1 \right]^n. \tag{A.4}$$

From (A.1)–(A.3) it follows that $g \in \mathcal{C}^1(\mathbb{R}^+)$ and

$$\forall t \geq 0, \quad \begin{cases} 1 \leq g_n'(t) \leq (C_2+1)n\big(g_n(t)\big)^{1+\eta/n}, \\ 1 \leq g_n(t) \leq \big(g_n'(t)\big)^{1+1/(n-1)}, \end{cases} \tag{A.5}$$

$$\forall t,s \in \mathbb{R}, \quad \big|\mathbf{A}(x,s) - \mathbf{A}(x,t)\big| \leq \left| \int_s^t g_n(|z|)\,\mathrm{d}z \right|. \tag{A.6}$$

Let us emphasize that we can choose n such that η/n and $1/(n-1)$ are small enough.
Step 2. Let $\mu > 0$ and from Step 1 let $\psi \in \mathcal{C}^1(\mathbb{R}^+)$ such that

$$\forall t \geq 0, \quad \begin{cases} 1 \leq \psi'(t) \leq M_1\big(\psi(t)\big)^{1+\mu}, \\ 1 \leq \psi(t) \leq \big(\psi'(t)\big)^{1+\mu}, \end{cases} \tag{A.7}$$

where $M_1 > 0$ and

$$\forall t,s \in \mathbb{R}, \quad \big|\mathbf{A}(x,s) - \mathbf{A}(x,t)\big| \leq \left| \int_s^t \psi(|z|)\,\mathrm{d}z \right|. \tag{A.8}$$

Let us define $\widetilde{\psi}(t) = \int_0^t \psi(z)\,\mathrm{d}z$, $\forall t \geq 0$ and $\varphi(t) = \Big(\big(\widetilde{\psi}(|t|) + 1\big)^3 - 1\Big)\mathrm{sign}(t)$, $\forall t \in \mathbb{R}$.

We now claim that if μ is chosen small enough then $\mathbf{A}(x,s)$ verifies (3.2) with the function φ.

We first derive some properties on φ and $\widetilde{\psi}$. The regularity of ψ implies that $\varphi \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^*)$ and

$$\forall t \in \mathbb{R}, \quad \varphi'(t) = 3\psi(|t|)\big(\widetilde{\psi}(|t|)+1\big)^2 \geq 1,$$
$$\forall t \neq 0, \quad \varphi''(t) = 3\big(\widetilde{\psi}(|t|)+1\big)\big(2\psi^2(t) + \psi'(t)\big(\widetilde{\psi}(t)+1\big)\big)\mathrm{sign(t)}.$$

From (A.7) and since ψ is an increasing function it follows that $\forall t > 0$,

$$\begin{aligned} 1 \leq \varphi''(t) &\leq 6\big(\widetilde{\psi}(|t|)+1\big)\psi(t)\Big(\int_0^t \psi'(z)\,\mathrm{d}z + \psi(0)\Big) + 3\big(\psi(t)\big)^{1+\mu} M_1\big(\widetilde{\psi}(|t|)+1\big)^2 \\ &\leq 6\big(\widetilde{\psi}(t)+1\big)\psi(t)\Big(M_1\psi^\mu(t)\int_0^t \psi(z)\,\mathrm{d}z + \psi(0)\Big) + 3M_1\big(\varphi'(t)\big)^{1+\mu} \\ &\leq 6\big(\psi(t)\big)^{1+\mu} M_1\big(\widetilde{\psi}(t)+1\big)\big(\widetilde{\psi}(t)+\psi(0)\big) + 3M_1\big(\varphi'(t)\big)^{1+\mu} \\ &\leq M_2\big(\varphi'(t)\big)^{1+\mu}, \end{aligned} \tag{A.9}$$

where M_2 is a positive constant depending on M_1 and $\psi(0)$. Moreover inequalities (A.7) give also that for any $t > 0$

$$\psi(t)^{1-\mu} \leq M_3\big(\widetilde{\psi}(t)+1\big), \tag{A.10}$$

$$\widetilde{\psi}(t) \leq M_4\big(\psi(t)\big)^{1+\mu+\mu^2}, \tag{A.11}$$

where M_3 and M_4 are positive constants.

Let $0 < K \leq K_0$ (K_0 will be chosen later) and r, s be two real numbers such that $\big|\varphi(s) - \varphi(r)\big| \leq K$. To prove condition (3.2) let us write

$$\left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,r)}{\varphi'(r)}\right| \leq \frac{1}{\varphi'(s)}\big|\mathbf{A}(x,s) - \mathbf{A}(x,r)\big| + \big|\mathbf{A}(x,r)\big|\left|\frac{\varphi'(r)-\varphi'(s)}{\varphi'(r)\varphi'(s)}\right|. \tag{A.12}$$

We first deal with the case $rs > 0$. Without loss of generality we assume that $0 < s < r$ (the case $r < s < 0$ is obtained by symmetry). From (A.8) and (A.12) we have

$$\left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,r)}{\varphi'(r)}\right| \leq \frac{1}{\varphi'(r)}\int_s^r \psi(z)\,\mathrm{d}z + \big|\mathbf{A}(x,r)\big|\frac{\int_s^r \varphi''(z)\,\mathrm{d}z}{\varphi'(r)\varphi'(s)}$$

Gathering (A.7)–(A.11) together with standard analysis arguments and a calculus lead to

$$\begin{aligned} \left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,r)}{\varphi'(r)}\right| &\leq \frac{1}{3\varphi'(r)(1+\widetilde{\psi}(s))^2}\int_s^r \varphi'(z)\,\mathrm{d}z + |\mathbf{A}(x,r)|\frac{M_2\varphi'(r)^\mu}{\varphi'(r)\varphi'(s)}\int_s^r \varphi'(z)\,\mathrm{d}z \\ &\leq \frac{K}{3\varphi'(r)(1+\varphi(s))^{2/3}} + \big(\|\mathbf{A}(x,0)\|_{L^\infty(\Omega)} + \widetilde{\psi}(r)\big)\frac{M_2 K}{\varphi'(r)^{1-\mu}\varphi'(s)} \\ &\leq \frac{M_5 K}{\varphi'(r)(1+\varphi(r)+\varphi(s))^{2/3}} + \frac{M_6 K}{\psi(r)^{1-\mu}\big(1+\widetilde{\psi}(r)\big)^{1-2\mu}\varphi'(s)} \\ &\leq \frac{M_5 K}{\varphi'(r)(1+\varphi(r)+\varphi(s))^{2/3}} + \frac{M_7 K}{\big(1+\widetilde{\psi}(r)\big)^{2-2\mu-\frac{2\mu+\mu^2}{1+\mu+\mu^2}}\varphi'(s)} \\ &\leq \frac{M_5 K}{\varphi'(r)(1+\varphi(r)+\varphi(s))^{2/3}} + \frac{M_8 K}{(1+\varphi(r)+\varphi(s))^{\frac{1}{3}\left(2-2\mu-\frac{2\mu+\mu^2}{1+\mu+\mu^2}\right)}\varphi'(s)}. \end{aligned}$$

We are now in a position to choose $\mu > 0$ such that

$$\frac{1}{2} < \delta \overset{\text{def}}{=} \frac{1}{3}\big(2 - 2\mu - \frac{2\mu+\mu^2}{1+\mu+\mu^2}\big) < \frac{2}{3}.$$

Then we have

$$\left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,r)}{\varphi'(r)}\right| \le \frac{M_5 K}{\varphi'(r)(1+\varphi(r)+\varphi(s))^\delta} + \frac{M_8 K}{(1+\varphi(r)+\varphi(s))^\delta \varphi'(s)}. \tag{A.13}$$

Let $K_0 > 0$ such that $M_2 K_0 < 1/2$. Then we have

$$0 \le \varphi'(r) - \varphi'(s) \le \int_s^r \varphi''(z)\,\mathrm{d}z \le M_2\big(\varphi'(r)\big)^\mu K \le M_2 K_0\big(\varphi'(r)\big)^\mu \le \frac{1}{2}\varphi'(r). \tag{A.14}$$

At last, inequalities (A.13) and (A.14) yield that

$$\left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,r)}{\varphi'(r)}\right| \le \frac{M_9 K}{(\varphi'(r))^{1/2}(\varphi'(r))^{1/2}(1+\varphi(r)+\varphi(s))^\delta}, \tag{A.15}$$

where M_9 is a positive constant independent of t, s and K, that is (3.2) when $rs > 0$

When $rs \le 0$, since $\varphi' \ge 1$ we have $|r| \le K \le K_0$ and $|s| \le K \le K_0$. Since $\varphi' \in \mathcal{C}^1(\mathbb{R}^*) \cap C(\mathbb{R})$, (A.8) and (A.12) give that there exists $M_{10} > 0$ (independent of r, s and K) such that

$$\left|\frac{\mathbf{A}(x,s)}{\varphi'(s)} - \frac{\mathbf{A}(x,r)}{\varphi'(r)}\right| \le \frac{M_{10} K}{\varphi'(s)^{1/2}\varphi'(r)^{1/2}\big(1+|\varphi(s)|+|\varphi(r)|\big)^\delta}, \tag{A.16}$$

that is (3.2) for $rs \le 0$. The proof of Lemma A.1 is complete. □

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