Existence and stability results for renormalized solutions to noncoercive nonlinear elliptic equations with measure data

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Abstract In this paper we prove the existence of a renormalized solution to a class of nonlinear elliptic problems whose prototype is

$$(P)\qquad \begin{cases} -\triangle_p u - \operatorname{div}(c(x)|u|^\gamma) + b(x)|\nabla u|^\lambda = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\triangle_p$ is the so called $p-$Laplace operator, $1 < p < N$, μ is a Radon measure with bounded variation on Ω, $0 \leq \gamma \leq p-1$, $0 \leq \lambda \leq p-1$, $|c|$ and b belong to the Lorentz spaces $L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$ and $L^{N,1}(\Omega)$ respectively. In particular we prove the existence result under the assumption that $\gamma = \lambda = p-1$, $\|b\|_{L^{N,1}(\Omega)}$ is small enough and $|c| \in L^{\frac{N}{p-1},r}(\Omega)$, with $r < +\infty$. We also prove a stability result for renormalized solutions to a class of noncoercive equations whose prototype is (P) with $b \equiv 0$.

Key Words: Existence, stability, nonlinear elliptic equations, noncoercive problems, measures data, renormalized solutions.

1 Introduction

In this paper we consider nonlinear elliptic problems whose prototype is

$$\begin{cases} -\triangle_p u - \operatorname{div}(c(x)|u|^\gamma) + b(x)|\nabla u|^\lambda = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\triangle_p$ is the so called $p-$Laplace operator, p is a real number such that $1 < p < N$, μ is a Radon measure with bounded variation on Ω, $0 \leq \gamma \leq p-1$, $0 \leq \lambda \leq p-1$, $|c|$ and b belong to the Lorentz spaces $L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$ and $L^{N,1}(\Omega)$, respectively.

We are interested in existence results for renormalized solution to (1.1).

The difficulties which arise in proving existence results for solutions to (1.1) are due both to the lack of coercivity of the operator and to the right-hand side which is a measure (and not an element of the dual space $W^{-1,p'}(\Omega)$).

Existence results for noncoercive elliptic problems are well-known when the datum μ belongs to the dual space $W^{-1,p'}(\Omega)$. Indeed the linear case, i. e. $p = 2$, $\gamma = \lambda = 1$, has been studied by Stampacchia in [St] (see also [Dr]), the case where the operator has only the term $b(x)|\nabla u|^\lambda$ is studied in [DP], the case where the operator has only the term $-\text{div}(c(x)|u|^\gamma)$ is studied in [B] and finally the case where the operator has the two lower order terms $-\text{div}(c(x)|u|^\gamma)$ and $b(x)|\nabla u|^\lambda$ is studied in [DPo2] (see also [G2] for a different proof).

If p is greater than the dimension N of the ambient space, then, by Sobolev embedding theorem and duality arguments, the space of measures with bounded variation on Ω is a subset of $W^{-1,p'}(\Omega)$, so that the existence of solutions in $W_0^{1,p}(\Omega)$ is a consequence of previous results. Thus this explain our restriction on p.

However, when $p \leq N$, it is necessary to change the framework in order to study problem (1.1), since simple examples (the Laplace operator in a ball, i.e. $p = 2$, $b = 0$, $c = 0$, and μ the Dirac mass in the center) show that, in general, the solution has not to be expected in the energy space $W_0^{1,p}(\Omega)$.

In the linear case Stampacchia defined a notion of solution of problem (1.1) by "duality" ([St]), for which he proved the existence and uniqueness under the assumption that 0 is not in the spectrum of the operator, condition which is satisfied if, for example, $\|c\|_{L^{\frac{N}{p-1}}(\Omega)}$ or $\|b\|_{L^N(\Omega)}$ is small enough. He also proved that such a solution satisfies the equation in distributional sense and it belongs to $W_0^{1,q}(\Omega)$ for every $q < N/(N-1)$. The techniques used by Stampacchia heavily relies on a duality argument, so that they can not be extended to the general nonlinear case, except in the case where $p = 2$, the operator has not lower order terms and it is Lipschitz continuous with respect to the gradient ([M2]).

The nonlinear case was firstly studied in [BG1], [BG2], then the effect of lower order terms were analyzed in [D] (where a term $b(x)|\nabla u|^\lambda$ is considered) and in [DPo1] (where both terms $-\text{div}(c(x)|u|^\gamma)$ and $b(x)|\nabla u|^\lambda$ are considered); in all these papers the existence of a solution which belongs to $W_0^{1,q}(\Omega)$ for every $q < N(p-1)/(N-1)$ and satisfies the equation in the distributional sense is proved when $p > 2 - \frac{1}{N}$. The hypothesis on p is motivated by the fact that, if $p \leq 2 - \frac{1}{N}$, then $\frac{N(p-1)}{N-1} \leq 1$. On the other hand a classical counterexample ([S], see also [P]) shows that in general such a solution is not unique.

This implies that, in order to obtain the existence and uniqueness of a solution for p close to 1, i.e. $p \leq 2 - \frac{1}{N}$, it is necessary to go out of the framework of classical Sobolev spaces.

For this reason two equivalent notions of solutions have been introduced: the notion of entropy solution in [BBGGPV], [BGO] and the notion of renormalized solution in [LM], [M1], for which the existence and uniqueness have been proved in the case where the datum μ belongs to $L^1(\Omega)$ or to $L^1(\Omega) + W^{-1,p'}(\Omega)$. In [DMOP] the notion of renormalized solution has been extended to the case of general measure with bounded total variation and existence (and partial uniqueness results) is proved. In such papers operators without lower order terms are considered.

Both difficulties (right-hand side measure and lower order terms, which produce a lack of coercivity) have been faced in [B], [BGu1], [BGu2] and [BMMP3]. In [B] the existence of entropy solutions is proved when the datum μ belongs to $L^1(\Omega)$ and the operator has a lower order term of the type $-\text{div}\ (c(x)|u|^\gamma)$; in [BGu1] and [BGu2] the existence of a renormalized solution is proved in the same case. Finally in [BMMP3] the existence of a renormalized solution is proved when the datum μ is a general measure with bounded total variation and the operator has only a lower order term of the type $b(x)|\nabla u|^\lambda$. In such papers no assumptions on the smallness of the coefficients are made and therefore the operators are in general noncoercive.

Uniqueness results for renormalized solution are proved in [BMMP2], when the datum μ is a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$ and the operator has a lower order term of the type $b(x)|\nabla u|^\lambda$ (see [BMMP4] for the case where μ belongs to $W^{-1,p'}(\Omega)$) and in [BGu1], [BGu2] when μ is a measure in $L^1(\Omega)$ and a lower order term of the type $-\text{div}(c(x)|u|^\gamma)$ is considered (see also [G1] for further uniqueness results).

In the present paper and in [GM], we prove the existence of renormalized solutions for the problems whose prototype is (1.1), where both the two lower terms $-\text{div}(c(x)|u|^\gamma)$ and $b(x)|\nabla u|^\lambda$ appear and where $0 \leq \gamma \leq p-1$, $0 \leq \lambda \leq p-1$, $|c|$ belongs to the Lorentz space $L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$, b belongs to Lorentz space $L^{N,1}(\Omega)$ and μ is a Radon measure with bounded variation on Ω. In both papers we do not make any coercivity assumption on the operator: we assume that the norm of one of the two coefficients is small when $\gamma = \lambda = p-1$, while no smallness of such norms is required when γ or λ are less than $p-1$.

In the present paper we consider nonlinear elliptic problems whose model is (1.1) and we prove an existence result in the case where μ is a Radon measure with bounded variation on Ω, $\gamma = p-1$, $\lambda = p-1$, $\|b\|_{L^{N,1}(\Omega)}$ is small enough and $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $\frac{N}{p-1} \leq r < +\infty$ is large. The case $\gamma = p-1$ and $\lambda < p-1$ and the case $\gamma < p-1$ and $\lambda < p-1$ are also studied.

The counterpart of the existence result proved in the present paper can be found in [GM], where in particular we prove the existence of a renormalized solution for the problem (1.1) in the case where μ is a Radon measure with bounded variation on Ω, $\gamma = \lambda = p-1$, $\|b\|_{L^{N,1}(\Omega)}$ is large and $\|c\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $\frac{N}{p-1} \le r < +\infty$ is small enough.

Let us now explain the idea of the proof of the existence result in the present paper. The first difficulty is to obtain some a priori estimate for $|\nabla u|^{p-1}$. This is done by proving uniform estimates of the level sets of $|u|$ (cf. [B], [BGu1], [BGu2]), which allow to obtain an estimate of $\nabla T_k(u)$ of the type $\|\nabla T_k(u)\|^p_{(L^p(\Omega))^N} \le kM + L$ for every $k > 0$. Such estimate of $\nabla T_k(u)$ then imply $\||\nabla u|^{p-1}\|_{(L^{N',\infty}(\Omega))^N} \le c$, thanks to a generalization of a result of [BBGGPV], proved in [BMMP3]. Finally we use an extension of the stability result proved in [DMOP] (see also [MP] and [M]). It allows us to handle the term $-\operatorname{div}(c(x)|u|^\gamma)$, which in general does not belong to the dual space $W^{-1,p'}(\Omega)$. Such a result could be proved by using the same arguments of [DMOP], but actually the proof which we give here is slightly different.

Finally we explicitly remark that, as for the existence result proved in the present paper, the main difficulty in proving the existence result of [GM] is to obtain some a priori estimate for $|\nabla u|^{p-1}$. Such a priori estimates are obtain by using a different method.

2 Definitions and statement of existence result

In this section we recall some well-known results about the decomposition and convergence of measures (cf. [DMOP]) and some properties of Lorentz spaces (see e.g. [Lo], [H], [O]), which we will use in the following. Then we give the definition of a renormalized solution to nonlinear elliptic problems whose right-hand side is a Radon measure (cf. [DMOP]) and we state our existence result.

2.1 Preliminaries about measures

In this paper Ω is a bounded open subset of $\mathbb{R}^N$, $N \ge 2$, and p is a real number, $1 < p < N$, with p' defined by $1/p + 1/p' = 1$.

We denote by $M_b(\Omega)$ the space of Radon measures on Ω with bounded variation and by $C^0_b(\Omega)$ the space of bounded, continuous functions on Ω. Moreover μ^+ and μ^- denote the positive and the negative parts of the measure μ, respectively.

Definition 2.1 We say that a sequence $\{\mu_\varepsilon\}$ of measures in $M_b(\Omega)$ converges in the narrow topology to a measure μ in $M_b(\Omega)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi d\mu_\varepsilon = \int_\Omega \varphi d\mu, \tag{2.1}$$

for every $\varphi \in C^0_b(\Omega)$.

Remark 2.2 We recall that, if μ_ε is a nonnegative measure in $M_b(\Omega)$, then $\{\mu_\varepsilon\}$ converges in the narrow topology to a measure μ if and only if $\mu_\varepsilon(\Omega)$ converges to $\mu(\Omega)$ and (2.1) holds true for every $\varphi \in C_0^\infty(\Omega)$. It follows that if μ_ε is a nonnegative measure, μ_ε converges in the narrow topology to μ if and only if (2.1) holds true for any $\varphi \in C^\infty(\bar{\Omega})$.

We denote by $\text{cap}_p(B, \Omega)$ the standard capacity defined from $W_0^{1,p}(\Omega)$ of a Borel set B and we define $M_0(\Omega)$ as the set of the measures μ in $M_b(\Omega)$ which are absolutely continuous with respect to the p-capacity, i.e. which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\text{cap}_p(B, \Omega) = 0$. We define $M_s(\Omega)$ as the set of all the measures μ in $M_b(\Omega)$ which are singular with respect to the p-capacity, i.e. which are concentrated in a set $E \subset \Omega$ such that $\text{cap}_p(E, \Omega) = 0$.

The following result allows to split every measure in $M_b(\Omega)$ with respect to the p-capacity ([FST], Lemma 2.1).

Proposition 2.3 *For every measure μ in $M_b(\Omega)$ there exists an unique pair of measures (μ_0, μ_s), with $\mu_0 \in M_0(\Omega)$ and $\mu_s \in M_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$.*

The measures μ_0 and μ_s will be called the absolutely continuous part and the singular part of μ with respect to the p-capacity. Actually, for what concerns μ_0 one has the following decomposition result ([BGO], Theorem 2.1)

Proposition 2.4 *Let μ_0 be a measure in $M_b(\Omega)$. Then μ_0 belongs to $M_0(\Omega)$ if and only if it belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Thus if μ_0 belongs to $M_0(\Omega)$, there exists f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$ such that*

$$\mu_0 = f - \text{div}(g),$$

in the sense of distributions. Moreover every function $v \in W_0^{1,p}(\Omega)$ is measurable with respect to μ_0 and belongs to $L^\infty(\Omega, \mu_0)$ if v further belongs to $L^\infty(\Omega)$, and one has

$$\int_\Omega v d\mu_0 = \int_\Omega f v + \int_\Omega g \nabla v, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

As a consequence of the previous results and the Hahn decomposition Theorem we get the following result

Proposition 2.5 *Every measure μ in $M_b(\Omega)$ can be decomposed as follows*

$$\mu = \mu_0 + \mu_s = f - \text{div}(g) + \mu_s^+ - \mu_s^-,$$

where μ_0 is a measure in $M_0(\Omega)$, hence can be written as $f - \text{div}(g)$, with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, and where μ_s^+ and μ_s^- (the positive and the negative parts of μ_s) are two nonnegative measures in $M_s(\Omega)$, which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity.

Finally we recall the following result which will be used several times in Section 4 to prove the stability result. It is a consequence of Egorov theorem.

Proposition 2.6 *Let Ω be a bounded open subset of $\mathbb{R}^N$. Assume that ρ_ε is a sequence of $L^1(\Omega)$ functions converging to ρ weakly in $L^1(\Omega)$ and assume that σ_ε is a sequence of $L^\infty(\Omega)$ functions which is bounded is $L^\infty(\Omega)$ and converges to σ almost everywhere in Ω. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\sigma_\varepsilon = \int_\Omega \rho\sigma.$$

2.2 Preliminaries about Lorentz spaces

For $1 < q < \infty$ and $1 < s < \infty$ the Lorentz space $L^{q,s}(\Omega)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,s}(\Omega)} = \left(\int_0^{|\Omega|} [f^*(t)t^{\frac{1}{q}}]^s \frac{dt}{t}\right)^{1/s} < +\infty, \tag{2.2}$$

endowed with the norm defined by (2.2).
Here f^* denotes the decreasing rearrangement of f, i.e. the decreasing function defined by

$$f^*(t) = \inf\{s \ge 0 \;:\; \text{meas}\,\{x \in \Omega \;:\; |f(x)| > s\} < t\}, \quad t \in [0, |\Omega|].$$

For references about rearrangements see, for example, [CR], [K].

For $1 < r < \infty$, the Lorentz space $L^{r,\infty}(\Omega)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{r,\infty}(\Omega)} = \sup_{t>0} t\,[\text{meas}\,\{x \in \Omega \;:\; |f(x)| > t\}]^{1/r} < +\infty, \tag{2.3}$$

endowed with the norm defined by (2.3).

Lorentz spaces are "intermediate spaces" between the Lebesgue spaces, in the sense that, for every $1 < s < r < \infty$, we have

$$L^{r,1}(\Omega) \subset L^{r,r}(\Omega) = L^r(\Omega) \subset L^{r,\infty}(\Omega) \subset L^{s,1}(\Omega). \tag{2.4}$$

The space $L^{r,\infty}(\Omega)$ is the dual space of $L^{r',1}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, and we have the generalized Hölder inequality

$$\begin{cases} \forall f \in L^{r,\infty}(\Omega),\ \forall g \in L^{r',1}(\Omega), \\ \displaystyle\int_\Omega |fg| \le \|f\|_{L^{r,\infty}(\Omega)} \|g\|_{L^{r',1}(\Omega)}. \end{cases} \tag{2.5}$$

More generally, if $1 < p < \infty$ and $1 \le q \le \infty$, we get

$$\begin{cases} \forall f \in L^{p_1,q_1}(\Omega), \ \forall g \in L^{p_2,q_2}(\Omega), \\ \\ \|fg\|_{L^{p,q}(\Omega)} \le \|f\|_{L^{p_1,q_1}(\Omega)} \|g\|_{L^{p_2,q_2}(\Omega)}, \\ \\ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \end{cases} \tag{2.6}$$

Improvements of classical Sobolev inequalities involving Lorentz spaces are proved, for example, in [ALT]. In the present paper we will only use the following generalized Sobolev inequality: a positive constant $S_{N,p}$ depending only on p and N exists such that

$$\|v\|_{L^{p^*,p}(\Omega)} \le S_{N,p} \|v\|_{W_0^{1,p}(\Omega)}, \tag{2.7}$$

for every $v \in W_0^{1,p}(\Omega)$.

2.3 Definition of renormalized solution and statement of existence result

In the present paper we consider a nonlinear elliptic problem which can formally be written as

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u) + K(x,u)) + H(x,u,\nabla u) + G(x,u) = \mu - \mathrm{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $a : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \longmapsto \mathrm{I\!R}^N$ and $K : \Omega \times \mathrm{I\!R} \longmapsto \mathrm{I\!R}^N$ are Carathéodory functions satisfying

$$a(x,s,\xi)\xi \ge \alpha|\xi|^p, \quad \alpha > 0, \tag{2.9}$$

$$|a(x,s,\xi)| \le c\left[|\xi|^{p-1} + |s|^{p-1} + a_0(x)\right], \quad a_0(x) \in L^{p'}(\Omega), \quad c > 0, \tag{2.10}$$

$$(a(x,s,\xi) - a(x,s,\eta), \xi - \eta) > 0, \quad \xi \ne \eta, \tag{2.11}$$

$$\begin{cases} |K(x,s)| \le c_0(x)|s|^\gamma + c_1(x), \\ \\ 0 \le \gamma \le p-1, \quad c_0 \in L^{\frac{N}{p-1},r}(\Omega), \quad \frac{N}{p-1} \le r \le +\infty, \quad c_1 \in L^{p'}(\Omega), \end{cases} \tag{2.12}$$

for almost every $x \in \Omega$ and for every $s \in \mathrm{I\!R}$, $\xi \in \mathrm{I\!R}^N$, $\eta \in \mathrm{I\!R}^N$. Moreover $H : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \longmapsto \mathrm{I\!R}$ and $G : \Omega \times \mathrm{I\!R} \longmapsto \mathrm{I\!R}$ are Carathéodory functions satisfying

$$\begin{cases} |H(x,s,\xi)| \le b_0(x)|\xi|^\lambda + b_1(x), \\ \\ 0 \le \lambda \le p-1, \quad b_0 \in L^{N,1}(\Omega), \quad b_1 \in L^1(\Omega), \end{cases} \tag{2.13}$$

$$G(x,s)s \geq 0, \tag{2.14}$$

$$\begin{cases} |G(x,s)| \leq d_1(x)|s|^t + d_2(x), \\ \\ d_1 \in L^{z',1}(\Omega), \; d_2 \in L^1(\Omega), \end{cases} \tag{2.15}$$

for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, where

$$0 \leq t < \frac{N(p-1)}{N-p}, \quad z = \frac{N(p-1)}{N-p}\frac{1}{t} \text{ and } \frac{1}{z} + \frac{1}{z'} = 1. \tag{2.16}$$

Finally μ is a measure in $M_b(\Omega)$ that is decomposed in

$$\mu = f - \mathrm{div}(g) + \mu_s^+ - \mu_s^-, \tag{2.17}$$

according to Proposition 2.3, and

$$F \in \left(L^{p'}(\Omega)\right)^N. \tag{2.18}$$

Remark 2.7 Observe that, by (2.4) if the function c_0 belongs to the Lebesgue space $L^t(\Omega)$ for some $t \geq \frac{N}{p-1}$ and the function b_0 belongs to the Lebesgue space $L^q(\Omega)$ for some $q > N$, then the conditions $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r \leq +\infty$, and $b_0 \in L^{N,1}(\Omega)$ (as requested in hypotheses (2.12) and (2.13)) are satisfied.

In the present paper we consider renormalized solution to the problem (2.8). Before giving the definition of such a notion of solution, we need a few notation and definitions.

For $k > 0$, denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k, that is

$$T_k(s) = \begin{cases} s & |s| \leq k, \\ k \text{ sign}(s) & |s| > k. \end{cases}$$

Consider a measurable function $u : \Omega \to \bar{\mathbb{R}}$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists (see e.g. [BBGGPV], Lemma 2.1) an unique measurable function $v : \Omega \to \bar{\mathbb{R}}^N$, finite almost everywhere, such that

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \quad \text{almost everywhere in} \quad \Omega, \quad \forall k > 0. \tag{2.19}$$

We define the gradient ∇u of u as this function v, and denote $\nabla u = v$. Note that the previous definition does not coincide with the definition of the distributional gradient. However if $v \in (L^1_{loc}(\Omega))^N$, then $u \in W^{1,1}_{loc}(\Omega)$ and v is the distributional gradient of u. In contrast there are examples of functions $u \notin L^1_{loc}(\Omega)$ (and thus such that the gradient of u in the distributional sense is not defined) for which the gradient ∇u is defined in the previous sense (see Remarks 2.10 and 2.11, Lemma 2.12 and Example 2.16 in [DMOP]).

Definition 2.8 We say that a function $u : \Omega \longmapsto \bar{\mathbb{R}}$, measurable on Ω, almost everywhere finite, is a renormalized solution of (2.8) if it satisfies the following conditions

$$T_k(u) \in W_0^{1,p}(\Omega), \quad \forall k > 0; \tag{2.20}$$

$$|u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega); \tag{2.21}$$

$$|\nabla u|^{p-1} \quad \text{belongs to} \quad L^{N',\infty}(\Omega), \tag{2.22}$$

where ∇u is the gradient introduced in (2.19);

$$\lim_{n\to+\infty} \frac{1}{n} \int_{n<u<2n} a(x,u,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi d\mu_s^+, \tag{2.23}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{-2n<u<-n} a(x,u,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi d\mu_s^-, \tag{2.24}$$

for every $\varphi \in C_b^0(\Omega)$;

$$\lim_{n\to+\infty} \frac{1}{n} \int_{n<|u|<2n} |K(x,u)||\nabla u| = 0; \tag{2.25}$$

and finally

$$\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\cdot\nabla u\, h'(u)v + \int_\Omega a(x,u,\nabla u)\cdot\nabla v\, h(u)\\
&\quad + \int_\Omega K(x,u)\cdot\nabla u\, h'(u)v + \int_\Omega K(x,u)\cdot\nabla v\, h(u)\\
&\quad + \int_\Omega H(x,u,\nabla u)h(u)v + \int_\Omega G(x,u)h(u)v\\
&= \int_\Omega fh(u)v + \int_\Omega (g+F)\cdot\nabla u\, h'(u)v + \int_\Omega (g+F)\cdot\nabla v\, h(u),
\end{aligned} \tag{2.26}$$

for every $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, for all $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $h(u)v \in W_0^{1,p}(\Omega)$.

Since $h(u)v \in W_0^{1,p}(\Omega)$ and since $\operatorname{supp}(h) \subset [-2n, 2n]$ (for a suitable $n > 0$ depending on h), we can rewrite (2.26) as follows

$$\begin{aligned}
&\int_\Omega a(x,T_{2n}(u),\nabla T_{2n}(u))\cdot\nabla T_{2n}(u)\, h'(u)v + \int_\Omega a(x,T_{2n}(u),\nabla T_{2n}(u))\cdot\nabla v\, h(u)\\
&\quad + \int_\Omega K(x,T_{2n}(u))\cdot\nabla T_{2n}(u)\, h'(u)v + \int_\Omega K(x,T_{2n}(u))\cdot\nabla v\, h(u)\\
&\quad + \int_\Omega H(x,T_{2n}(u),\nabla T_{2n}(u))h(u)v + \int_\Omega G(x,T_{2n}(u))h(u)v\\
&= \int_\Omega fh(u)v + \int_\Omega (g+F)\cdot\nabla T_{2n}(u)\, h'(u)v + \int_\Omega (g+F)\cdot\nabla v\, h(u).
\end{aligned} \tag{2.27}$$

Let us observe that every integral in (2.27) is well defined thanks to the fact that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, h has compact support and the assumptions (2.9)-(2.18).

Remark 2.9 Observe that every renormalized solution u of (2.8) is such that

$$|a(x,u,\nabla u)| \in L^{N',\infty}(\Omega), \qquad |K(x,u)| \in L^{N',r}(\Omega), \quad \frac{N}{p-1} \le r \le +\infty,$$

$$G(x,u) \in L^1(\Omega) \quad \text{and} \quad H(x,u,\nabla u) \in L^1(\Omega),$$

thanks to the conditions (2.21) and (2.22), and the growth conditions (2.10), (2.12), (2.13) and (2.15) on a, K, H and G respectively.

Observe also that, since $p < N$, then $L^{p'}(\Omega) \subset L^{N',r}(\Omega)$, $\frac{N}{p-1} \le r \le +\infty$ and therefore the term $K(x,u)$ does not belong in general to $(L^{p'}(\Omega))^N$.

Moreover u is a solution of (2.8) in the distributional sense, that is u satisfies the following equality

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot\nabla\phi + \int_\Omega K(x,u)\cdot\nabla\phi + \int_\Omega H(x,u,\nabla u)\phi + \int_\Omega G(x,u)\phi \\ = \int_\Omega \phi d\mu + \int_\Omega F\cdot\nabla\phi,\end{aligned} \tag{2.28}$$

for all $\phi \in C_0^\infty(\Omega)$.
Indeed we take $\phi \in C_0^\infty(\Omega)$ and $h = h_n$ defined by

$$h_n(s) = \begin{cases} 0, & |s| > 2n \\ \frac{2n-|s|}{n}, & n < |s| \le 2n \\ 1, & |s| \le n, \end{cases} \tag{2.29}$$

in (2.26); then we let n tend to infinity and we obtain (2.28).

The main result of the present paper is the following existence result (in Section 4 we state and we prove a generalization of the stability result of [DMOP])

Theorem 2.10 *Under assumptions (2.9)-(2.18), there exists at least one renormalized solution u of (2.8) if one of the following conditions holds true*

1) $\gamma = \lambda = p-1$, $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ and $\|b_0\|_{L^{N,1}(\Omega)}$ is small enough;

2) $\gamma = p-1$, $\lambda < p-1$, $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$;

3) $\gamma < p-1$, $\lambda < p-1$, $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$.

Remark 2.11 The counterpart of such existence result, that is the case $\gamma = \lambda = p-1$, $b_0 \in L^{N,1}(\Omega)$ and $\|c_0\|_L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$, small enough and the case $\gamma < p-1$, $\lambda = p-1$ with $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$ and $b_0 \in L^{N,1}(\Omega)$ (without smallness hypothesis on the norm) are investigated in [GM].

Remark 2.12 We assume that $\gamma = p-1$ and c_0 belongs to $L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ in condition 1) and 2), while we assume that $\gamma < p-1$ and c_0 belongs to $L^{\frac{N}{p-1},\infty}(\Omega)$ in condition 3). Actually we prove Theorem 2.10, by using the stability result (Theorem 4.1) which holds true under the assumption that $\gamma < p-1$ and $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$ or under the assumption that $\gamma = p-1$ and $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$.

The proof of Theorem 2.10 is made by several steps. We begin by approximating the data of the problem (2.8). Then we obtain the a priori estimate for the gradients of the solutions u_ε to the approximate problems. We prove them in Section 3 below, Theorem 3.2. The last step in the proof of Theorem 2.10 consists to pass to the limit in the approximate problems. This is done by reconduce the problem to apply the Theorem 4.1, which is an extension of the stability result proved in [DMOP] when $K(x,s) \equiv 0$; our result allows to deal with the term $-\mathrm{div}(K_\varepsilon(x,u_\varepsilon))$ which is not in general bounded in $W^{-1,p'}(\Omega)$.

3 Approximate problems and a priori estimates

We begin this Section by approximating the data of the problem (2.8). Then we prove Thorem 3.2 below which gives the a priori estimates for the gradients of the solutions u_ε to the approximate problems. Let us explain our method in the most delicate case, that is $\gamma = \lambda = p-1$. We firstly prove an estimate of the level sets of the function $|u_\varepsilon|$, which allows us to choose a level set $\{|u_\varepsilon| > \sigma\}$ in such a way that $\|c_0\|_{L^{\frac{N}{p-1},r}(|u_\varepsilon|>\sigma)}$ is small enough. This allows us to obtain an estimate for $\|\nabla T_k(u_\varepsilon)\|_{(L^p(\Omega))^N}$ which implies the estimate of $|\nabla u_\varepsilon|^{p-1}$ thanks to the generalization of the result of [BBGGPV] proved in [BMMP3] and stated below.

Lemma 3.1 *Assume that Ω is an open subset of $\mathbb{R}^N$ with finite measure and that $1 < p < N$. Let u be a measurable function satisfying $T_k(u) \in W_0^{1,p}(\Omega)$, for every positive k, and such that*

$$\int_\Omega |\nabla T_k(u)|^p \le Mk + L, \quad \forall k > 0, \tag{3.1}$$

where M and L are given constants. Then $|u|^{p-1}$ belongs to $L^{\frac{p^*}{p},\infty}(\Omega)$, $|\nabla u|^{p-1}$ belongs to $L^{N',\infty}(\Omega)$ and

$$\||u|^{p-1}\|_{L^{\frac{p^*}{p},\infty}(\Omega)} \le C(N,p)\left[M + |\Omega|^{\frac{1}{p^*}} L^{\frac{1}{p'}}\right], \tag{3.2}$$

$$\||\nabla u|^{p-1}\|_{L^{N',\infty}(\Omega)} \le C(N,p)\left[M + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}} L^{\frac{1}{p'}}\right], \tag{3.3}$$

where $C(N,p)$ is a constant depending only on N and p and where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$.

Let us introduce the approximate problems.

The Radon measure with bounded variation μ can be decomposed as

$$\mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-,$$

where $f \in L^1(\Omega)$, $g \in \left(L^{p'}(\Omega)\right)^N$ and μ_s^+ and μ_s^- (the positive and the negative parts of μ_s) are two nonnegative measures in $M_b(\Omega)$ which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity, according to Proposition 2.5.

As in [DMOP] (cf. [BMMP3]), we approximate the measure μ by a sequence μ_ε defined as

$$\mu_\varepsilon = f_\varepsilon - \operatorname{div}(g) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus,$$

where

$$\begin{cases} f_\varepsilon \quad \text{is a sequence of } L^{p'}(\Omega) \text{ functions} \\ \text{that converges to } f \text{ weakly in } L^1(\Omega), \end{cases} \tag{3.4}$$

$$\begin{cases} \lambda_\varepsilon^\oplus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \\ \text{that converges to } \mu_s^+ \text{ in the narrow topology of measures,} \end{cases} \tag{3.5}$$

and

$$\begin{cases} \lambda_\varepsilon^\ominus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \\ \text{that converges to } \mu_s^- \text{ in the narrow topology of measures.} \end{cases} \tag{3.6}$$

Observe that μ_ε belongs to $W^{-1,p'}(\Omega)$.

Let us denote by

$$K_\varepsilon(x,s) = K(x, T_{1/\varepsilon}(s)), \tag{3.7}$$

$$H_\varepsilon(x,s,\xi) = T_{1/\varepsilon}(H(x,s,\xi)), \tag{3.8}$$

$$G_\varepsilon(x,s) = T_{1/\varepsilon}(G(x,s)). \tag{3.9}$$

Therefore, by assumptions (2.12)-(2.15), we have

$$|K_\varepsilon(x,s)| \le |K(x,s)| \le c_0(x)|s|^\gamma + c_1(x), \tag{3.10}$$

$$|K_\varepsilon(x,s)| \le c_0(x)\frac{1}{\varepsilon^\gamma} + c_1(x), \tag{3.11}$$

$$|H_\varepsilon(x,s,\xi)| \le |H(x,s,\xi)| \le b_0(x)|\xi|^\lambda + b_1(x), \tag{3.12}$$

$$|H_\varepsilon(x,s,\xi)| \le \frac{1}{\varepsilon}, \tag{3.13}$$

$$G_\varepsilon(x,s)s \ge 0, \tag{3.14}$$

$$|G_\varepsilon(x,s)| \le |G(x,s)| \le d_1(x)|s|^r + d_2(x), \tag{3.15}$$

$$|G_\varepsilon(x,s)| \le \frac{1}{\varepsilon}. \tag{3.16}$$

Let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be a weak solution of the following problem

$$\begin{cases} -\mathrm{div}(a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon))+H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)+G_\varepsilon(x,u_\varepsilon)=\mu_\varepsilon-\mathrm{div}(F) & \text{in } \Omega \\ u_\varepsilon=0. & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

i.e.

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega) \\ \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v+\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla v \\ \qquad +\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v+\int_\Omega G_\varepsilon(x,u_\varepsilon)v \\ \qquad =\int_\Omega f_\varepsilon v+\int_\Omega (g+F)\cdot\nabla v+\int_\Omega \lambda_\varepsilon^\oplus v-\int_\Omega \lambda_\varepsilon^\ominus v, \\ \forall v\in W_0^{1,p}(\Omega). \end{cases} \tag{3.18}$$

The existence of a solution u_ε of (3.18) is a well-known result (see e.g. [L], [DPo2], [G2]).

The main result of this Section is Theorem 3.2 below which gives an a priori estimate for $|\nabla u_\varepsilon|^{p-1}$.

Theorem 3.2 *Under the hypotheses of Theorem 2.10, every solution u_ε of (3.18) satisfies*

$$\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)}\le c, \tag{3.19}$$

$$\||u_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)}\le c, \tag{3.20}$$

where c is a positive constant which depends only on p, $|\Omega|$, N, α, $\|b_0\|_{L^{N,1}(\Omega)}$, $\|b_1\|_{L^1(\Omega)}$, $\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)}$, $\|c_1\|_{L^{p'}(\Omega)}$, $\|g\|_{(L^{p'}(\Omega))^N}$, $\|F\|_{(L^{p'}(\Omega))^N}$, $\sup_\varepsilon \|f_\varepsilon\|_{L^1(\Omega)}$, $\sup_\varepsilon \left(\lambda_\varepsilon^\oplus(\Omega)+\lambda_\varepsilon^\ominus(\Omega)\right)$ and on $(c_0)^*$ the decreasing rearrangement of c_0.

Proof of Theorem 3.2

Let us begin by proving Theorem 3.2 when assumption 1) in Theorem 2.10 is satisfied, i.e. $\gamma=\lambda=p-1$, $c\in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1}\le r<+\infty$ and $\|b\|_{L^{N,1}(\Omega)}$ is small enough.

First step. In this step we prove the estimate of the level sets of the functions $|u_\varepsilon|$ given by (3.39) below. It is performed through a "log-type" estimate on u_ε (cf. [B], [BOP], [BGu1], [BGu2]).

Define the function $\psi:\mathbb{R}\to\mathbb{R}$ by

$$\psi(s)=\int_0^s \frac{1}{(A^{p'/p}+|r|)^p}dr, \tag{3.21}$$

where A is a positive constant which will be specified later.

Observe that the following property of $\psi(s)$ holds true

$$|\psi(s)| \leq \frac{1}{A^{p'}}, \qquad \forall s \in \mathbb{R}\,. \tag{3.22}$$

Observe also that $\psi(s)$ is a Lipschitz function such that $\psi(0)=0$. Therefore, since $u_\varepsilon \in W_0^{1,p}(\Omega)$, the function $\psi(u_\varepsilon)$ belongs to $W_0^{1,p}(\Omega)$. This allows us to use $\psi(u_\varepsilon)$ as test function in (3.17). Then we get

$$\begin{aligned}
&\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\psi'(u_\varepsilon) + \int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\psi'(u_\varepsilon)\\
&\qquad + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\psi(u_\varepsilon) + \int_\Omega G_\varepsilon(x,u_\varepsilon)\psi(u_\varepsilon)\\
&\quad = \int_\Omega f_\varepsilon\psi(u_\varepsilon) + \int_\Omega (g+F)\cdot\nabla u_\varepsilon\psi'(u_\varepsilon) + \int_\Omega \lambda_\varepsilon^{\oplus}\psi(u_\varepsilon) - \int_\Omega \lambda_\varepsilon^{\ominus}\psi(u_\varepsilon).
\end{aligned} \tag{3.23}$$

Now we evaluate the various integrals in (3.23).

By the definition (3.21) of $\psi(s)$ and ellipticity condition (2.9), we obtain

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\psi'(u_\varepsilon) \geq \alpha\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p}. \tag{3.24}$$

Let us now estimate $\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot(u_\varepsilon)\psi'\nabla u_\varepsilon\right|$.

By the growth condition (3.10) on K_ε and Young's inequality, since $\frac{|u_\varepsilon|}{(A^{p'/p}+|u_\varepsilon|)} \leq 1$ and $(A^{p'/p}+|u_\varepsilon|)^p \geq A^{p'}$ we get

$$\begin{aligned}
&\left|\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\psi'(u_\varepsilon)\right|\\
&\leq \int_\Omega c_0|\psi'(u_\varepsilon)||u_\varepsilon|^{p-1}|\nabla u_\varepsilon| + \int_\Omega c_1|\psi'(u_\varepsilon)||\nabla u_\varepsilon|\\
&= \int_\Omega c_0|u_\varepsilon|^{p-1}\frac{|\nabla u_\varepsilon|}{(A^{p'/p}+|u_\varepsilon|)^p} + \int_\Omega c_1\frac{|\nabla u_\varepsilon|}{(A^{p'/p}+|u_\varepsilon|)^p}\\
&\leq \frac{3^{p'/p}}{p'\alpha^{p'/p}}\|c_0\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha}{3p}\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p}\\
&\quad + \frac{3^{p'/p}}{p'\alpha^{p'/p}}\int_\Omega \frac{c_1^{p'}}{(A^{p'/p}+|u_\varepsilon|)^p} + \frac{\alpha}{3p}\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p}\\
&\leq \frac{3^{p'/p}}{p'\alpha^{p'/p}}\left(\|c_0\|^{p'}_{L^{p'}(\Omega)} + \frac{1}{A^{p'}}\|c_1\|^{p'}_{L^{p'}(\Omega)}\right) + \frac{2\alpha}{3p}\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p}.
\end{aligned}$$

Moreover, since we assume that $p<N$, then $p' < \frac{N}{p-1}$. This implies, since $\frac{N}{p-1} \leq r$, that $L^{\frac{N}{p-1},r}(\Omega) \subset L^{p'}(\Omega)$ and by inequality (2.6), we get

$$\|c_0\|_{L^{p'}(\Omega)} \leq \|1\|_{L^{\frac{pN}{(p-1)(N-p)},t}(\Omega)}\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)},$$

i.e.

$$\|c_0\|_{L^{p'}(\Omega)} \le \frac{Np}{(p-1)(N-p)t}|\Omega|^{\frac{(p-1)(N-p)t}{Np}}\|c_0\|_{L^{\frac{N}{p-1},r}(\Omega)},$$

where t is defined by $\frac{1}{p'} = \frac{1}{t} + \frac{1}{r}$.

Therefore

$$\begin{aligned}
&\left| \int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon \psi'(u_\varepsilon) \right| \\
&\le \frac{3^{p'/p}}{p'\alpha^{p'/p}}\left[\left(\frac{Np}{(p-1)(N-p)t}\right)^{p'}|\Omega|^{\frac{(N-p)t}{N}}\|c_0\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} + \frac{1}{A^{p'}}\|c_1\|^{p'}_{L^{p'}(\Omega)}\right] \\
&\quad + \frac{2\alpha}{3p}\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p}.
\end{aligned} \tag{3.25}$$

Let us now estimate $\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\psi(u_\varepsilon)\right|$.

By the definition (3.21) of $\psi(s)$, the growth assumption (3.12) on H_ε, the property (3.22) of $\psi(s)$ and the generalized Hölder inequality (2.5), we have

$$\begin{aligned}
&\left| \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\psi(u_\varepsilon) \right| \\
&\quad \le \int_\Omega b_0|\nabla u_\varepsilon|^{p-1}\psi(u_\varepsilon) + \int_\Omega b_1\psi(u_\varepsilon) \\
&\quad \le \frac{1}{A^{p'}}\left[\int_\Omega b_0|\nabla u_\varepsilon|^{p-1} + \int_\Omega b_1\right] \\
&\quad \le \frac{1}{A^{p'}}\left[\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)}\right].
\end{aligned} \tag{3.26}$$

Moreover, by the "sign condition" (3.14) on G_ε, we get

$$\int_\Omega G_\varepsilon(x,u_\varepsilon)\psi(u_\varepsilon) \ge 0. \tag{3.27}$$

Finally, since $(A^{p'/p}+|u_\varepsilon|)^p \ge A^{p'}$, we have

$$\begin{aligned}
&\int_\Omega (g+F)\cdot\nabla u_\varepsilon \psi'(u_\varepsilon) \\
&\quad = \int_\Omega \frac{(g+F)\cdot\nabla u_\varepsilon}{(A^{p'/p}+|u_\varepsilon|)^p} \\
&\quad \le \frac{3^{p'/p}}{p'\alpha^{p'/p}A^{p'}}\|g+F\|^p_{(L^{p'}(\Omega))^N} + \frac{\alpha}{3p}\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p}
\end{aligned} \tag{3.28}$$

and, by (3.22), we also get

$$\int_\Omega f_\varepsilon \psi(u_\varepsilon) \le \frac{1}{A^{p'}}\|f_\varepsilon\|_{L^1(\Omega)}, \tag{3.29}$$

$$\left|\int_\Omega \lambda_\varepsilon^\oplus \psi(u_\varepsilon)\right| \le \frac{1}{A^{p'}}\lambda_\varepsilon^\oplus(\Omega), \tag{3.30}$$

$$\left|\int_\Omega \lambda_\varepsilon^\ominus \psi(u_\varepsilon)\right| \le \frac{1}{A^{p'}}\lambda_\varepsilon^\ominus(\Omega). \tag{3.31}$$

Combining (3.23)-(3.31), we get

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p} \\
&\le \frac{3^{p'/p}}{\alpha^{p'/p+1}}\left(\frac{Np}{(p-1)(N-p)t}\right)^{p'} |\Omega|^{\frac{(N-p)t}{N}} \|c_0\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} \\
&\quad + \frac{p'}{\alpha A^{p'}}\Bigg\{ \|b_0\|_{L^{N,1}(\Omega)} \||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \\
&\quad + \frac{3^{p'/p}}{p'\alpha^{p'/p}}(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}) + M_0\Bigg\},
\end{aligned} \tag{3.32}$$

where

$$M_0 = \|b_1\|_{L^1(\Omega)} + \sup_\varepsilon \|f_\varepsilon\|_{L^1(\Omega)} + \sup_\varepsilon \left[\lambda_\varepsilon^\oplus(\Omega) + \lambda_\varepsilon^\ominus(\Omega)\right]. \tag{3.33}$$

Define

$$\begin{aligned}
A =& 1 + \frac{p'}{\alpha}\Bigg\{ \|b_0\|_{L^{N,1}(\Omega)} \||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \\
&+ \frac{3^{p'/p}}{p'\alpha^{p'/p}}(\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}) + M_0\Bigg\}.
\end{aligned} \tag{3.34}$$

Observe that $A > 1$ and therefore $\frac{1}{A^{p'}} \le \frac{1}{A}$. This implies that

$$\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p} \le \frac{3^{p'/p}}{\alpha^{p'/p+1}}\left(\frac{Np}{(p-1)(N-p)t}\right)^{p'} |\Omega|^{\frac{(N-p)t}{N}} \|c_0\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} + 1. \tag{3.35}$$

On the other hand, by Poincaré inequality, we get

$$\begin{aligned}
\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(A^{p'/p}+|u_\varepsilon|)^p} &= \int_\Omega \left|\nabla \log(A^{p'/p}+|u_\varepsilon|)\right|^p \\
&= \int_\Omega \left|\nabla \log\left(1+\frac{|u_\varepsilon|}{A^{p'/p}}\right)\right|^p \\
&\ge c(N,p)\int_\Omega \left[\log\left(1+\frac{|u_\varepsilon|}{A^{p'/p}}\right)\right]^p.
\end{aligned} \tag{3.36}$$

Therefore for any $\eta > 0$, we have

$$\begin{aligned}\text{meas}\left\{|u_\varepsilon| \geq \eta A^{p'/p}\right\} &= \frac{1}{[\log(1+\eta)]^p} \int_{\{|u_\varepsilon| \geq \eta A^{p'/p}\}} [\log(1+\eta)]^p \\ &\leq \frac{1}{[\log(1+\eta)]^p} \int_{\{|u_\varepsilon| \geq \eta A^{p'/p}\}} \left[\log\left(1 + \frac{|u_\varepsilon|}{A^{p'/p}}\right)\right]^p \\ &\leq \frac{1}{[\log(1+\eta)]^p} \int_\Omega \left[\log\left(1 + \frac{|u_\varepsilon|}{A^{p'/p}}\right)\right]^p\end{aligned} \tag{3.37}$$

Denote

$$C^* = \frac{1}{c(N,p)} \left[\frac{3^{p'/p}}{\alpha^{p'/p+1}} \left(\frac{Np}{(p-1)(N-p)t}\right)^{p'} |\Omega|^{\frac{(N-p)t}{N}} \|c_0\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} + 1\right]. \tag{3.38}$$

Combining (3.35) -(3.37), we get

$$\text{meas}\left\{|u_\varepsilon| \geq \eta A^{p'/p}\right\} \leq \frac{C^*}{[\log(1+\eta)]^p},$$

for any $\eta > 0$, or, equivalently, for any $\nu > 0$

$$\text{meas}\left\{|u_\varepsilon| \geq \exp(C^*\nu) A^{p'/p}\right\} \leq \frac{1}{\nu^p}. \tag{3.39}$$

Second step. Using in (3.18) the test function $T_k(u_\varepsilon)$, we obtain

$$\begin{aligned}&\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla T_k(u_\varepsilon) + \int_\Omega K_\varepsilon(x, u_\varepsilon) \cdot \nabla T_k(u_\varepsilon) \\ &\quad + \int_\Omega H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) + \int_\Omega G_\varepsilon(x, u_\varepsilon) T_k(u_\varepsilon) \\ &= \int_\Omega f_\varepsilon T_k(u_\varepsilon) + \int_\Omega (g+F) \cdot \nabla T_k(u_\varepsilon) \\ &\quad + \int_\Omega \lambda_\varepsilon^\oplus T_k(u_\varepsilon) - \int_\Omega \lambda_\varepsilon^\ominus T_k(u_\varepsilon).\end{aligned} \tag{3.40}$$

Now we evaluate the various terms in (3.40).

By ellipticity condition (2.9), we obtain

$$\begin{aligned}\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla T_k(u_\varepsilon) &= \int_{\{|u_\varepsilon| \leq k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \\ &\geq \alpha \int_\Omega |\nabla T_k(u_\varepsilon)|^p.\end{aligned} \tag{3.41}$$

Let us now estimate $\left|\int_\Omega K_\varepsilon(x, u_\varepsilon) \cdot \nabla T_k(u_\varepsilon)\right|$.

Denote

$$\sigma = \exp(C^* \nu) A^{p'/p}, \tag{3.42}$$

where ν is a positive constant which will be specified later.

Since by the generalized Sobolev inequality (2.7), $T_k(u_\varepsilon)$ belongs to $L^{p^*,p}(\Omega)$, then $T_k(u_\varepsilon)$ belongs also to $L^{p^*,t}(\Omega)$ for any $p \le t \le +\infty$. Moreover, by the growth condition (3.10) on K_ε, the generalized Hölder inequality (2.5) and the Young inequality, we get

$$\begin{aligned}
&\left| \int_\Omega K_\varepsilon(x, u_\varepsilon) \cdot \nabla T_k(u_\varepsilon) \right| \\
&\quad \le \int_\Omega c_0 |u_\varepsilon|^{p-1} |\nabla T_k(u_\varepsilon)| + \int_\Omega c_1 |\nabla T_k(u_\varepsilon)| \\
&\quad = \int_{|u_\varepsilon| \ge \sigma} c_0 |T_k(u_\varepsilon)|^{p-1} |\nabla T_k(u_\varepsilon)| + \int_{|u_\varepsilon| < \sigma} c_0 |T_k(u_\varepsilon)|^{p-1} |\nabla T_k(u_\varepsilon)| + \int_\Omega c_1 |\nabla T_k(u_\varepsilon)| \\
&\quad \le \|c_0\|_{L^{\frac{N}{p-1},r}(|u_\varepsilon| \ge \sigma)} \|T_k(u_\varepsilon)\|^{p-1}_{L^{p^*,t}(\Omega)} \|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\qquad + \frac{3^{p'/p}}{p' \alpha^{p'/p}} \sigma^p \|c_0\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha}{3p} \|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\qquad + \frac{3^{p'/p}}{p' \alpha^{p'/p}} \|c_1\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha}{3p} \|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N}. \\
&\quad \le C(N, p, |\Omega|) \|c_0\|_{L^{\frac{N}{p-1},r}(|u_\varepsilon| \ge \sigma)} \|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} + \frac{2\alpha}{3p} \|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\qquad + \frac{3^{p'/p}}{p' \alpha^{p'/p}} \left(\sigma^p \|c_0\|^{p'}_{L^{p'}(\Omega)} + \|c_1\|^{p'}_{L^{p'}(\Omega)} \right),
\end{aligned} \tag{3.43}$$

where t is choosed such that $\frac{1}{r} + \frac{p-1}{t} + \frac{1}{p} = 1$ and where $C(N, p, |\Omega|)$ is a positive constant which depend only on N, p and $|\Omega|$.

Let us now estimate $\left| \int_\Omega H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) \right|$.

By the growth assumption (3.12) on H_ε and the generalized Hölder inequality (2.5), we have

$$\begin{aligned}
&\left| \int_\Omega H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) \right| \\
&\qquad \le k \left[\|b_0\|_{L^{N,1}(\Omega)} \| |\nabla u_\varepsilon|^{p-1} \|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)} \right].
\end{aligned} \tag{3.44}$$

Moreover, by the "sign condition" (3.14) on G_ε, we get

$$\int_\Omega G_\varepsilon(x, u_\varepsilon) T_k(u_\varepsilon) \ge 0. \tag{3.45}$$

Finally

$$\int_\Omega f_\varepsilon T_k(u_\varepsilon) \le k \|f_\varepsilon\|_{L^1(\Omega)}, \tag{3.46}$$

$$\int_\Omega (g+F)\cdot\nabla T_k(u_\varepsilon) \le \frac{\alpha}{3p}\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} + \frac{3^{p'/p}}{p'\alpha^{p'/p}}\|g+F\|^{p'}_{(L^{p'}(\Omega))^N}, \tag{3.47}$$

$$\left|\int_\Omega \lambda_\varepsilon^\oplus T_k(u_\varepsilon)\right| \le k\lambda_\varepsilon^\oplus(\Omega), \tag{3.48}$$

$$\left|\int_\Omega \lambda_\varepsilon^\ominus T_k(u_\varepsilon)\right| \le k\lambda_\varepsilon^\ominus(\Omega). \tag{3.49}$$

Combining (3.40)-(3.49), for any $k>0$, we get

$$\begin{aligned}
&\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\quad \le \frac{p'}{\alpha}C(N,p,|\Omega|)\|c_0\|_{L^{\frac{N}{p-1},r}(|u_\varepsilon|\ge\sigma)}\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \\
&\quad\quad + \frac{p'}{\alpha}k\left[\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} + M_0\right] \\
&\quad\quad + \frac{3^{p'/p}}{\alpha^{p'/p+1}}\left(\sigma^p\|c_0\|^{p'}_{L^{p'}(\Omega)} + \|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}\right),
\end{aligned}$$

where M_0 is defined by (3.33).

On the other hand, since (3.39) holds true, we can choose $\nu=\bar\nu$ in such a way that

$$\text{meas}\left\{|u_\varepsilon| \ge \exp(C_1\bar\nu)A^{p'/p}\right\} \le \frac{1}{\bar\nu^p} < \tau,$$

for some $\tau>0$, implies

$$\frac{p'}{\alpha}C(N,p,|\Omega|)\|c_0\|_{L^{\frac{N}{p-1},r}(|u_\varepsilon|\ge\exp(C_1\bar\nu))} < \frac{1}{2}.$$

Observe that such $\bar\nu$ is independent on ε.
Therefore we obtain

$$\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \le M^*k + L^*, \qquad \forall k>0, \tag{3.50}$$

where

$$M^* = \frac{2p'}{\alpha}\left[\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} + M_0\right],$$

$$L^* = 2\frac{3^{p'/p}}{\alpha^{p'/p+1}}(\sigma^p\|c_0\|^{p'}_{L^{p'}(\Omega)} + \|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N})$$

and M_0 is defined by (3.33).

By Lemma 3.1, the definition (3.42) of σ and the definition (3.34) of A, we get

$$
\begin{aligned}
&\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \le C(N,p)\left[M^* + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}}(L^*)^{\frac{1}{p'}}\right] \\
&\le C(N,p)\left\{2\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} + 2\frac{p'}{\alpha}M_0\right. \\
&\left.+|\Omega|^{\frac{1}{N'}-\frac{1}{p'}}\left[\frac{3^{p'-1+1/p}2^{1/p'}}{\alpha}\left(\exp(\frac{pC^*\bar\nu}{p'})A\|c_0\|_{L^{p'}(\Omega)} + \|c_1\|_{L^{p'}(\Omega)} + \|g+F\|_{(L^{p'}(\Omega))^N}\right)\right]\right\},
\end{aligned}
$$

and then

$$
\begin{aligned}
&\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \le C^{**} \\
&+\frac{p'}{\alpha}C(N,p)\left(2+|\Omega|^{\frac{1}{N'}-\frac{1}{p'}}\frac{3^{p'-1+1/p}2^{1/p'}}{\alpha}\exp(\frac{pC^*\bar\nu}{p'})\|c_0\|_{L^{p'}(\Omega)}\right) \\
&\times\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)},
\end{aligned}
\tag{3.51}
$$

with

$$
\begin{aligned}
C^{**} = C(N,p)2\frac{p'}{\alpha}M_0 + C(N,p)|\Omega|^{\frac{1}{N'}-\frac{1}{p'}}\Bigg[&\frac{3^{p'-1+1/p}2^{1/p'}}{\alpha}\exp(\frac{pC^*\bar\nu}{p'})\|c_0\|_{L^{p'}(\Omega)} \\
&\times\left(1+\frac{3^{p'/p}}{p'\alpha^{p'/p}}(\|c_1\|_{L^{p'}(\Omega)} + \|g+F\|_{(L^{p'}(\Omega))^N})\right. \\
&\left.+M_0\right) + \\
&\|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}\Bigg]
\end{aligned}
$$

Since we assume that $\|b_0\|_{L^{N,1}(\Omega)}$ is small enough, or more exactly,

$$
\|b_0\|_{L^{N,1}(\Omega)} < \frac{\alpha}{p'C(N,p)\left(2+|\Omega|^{\frac{1}{N'}-\frac{1}{p'}}\frac{3^{p'-1+1/p}2^{1/p'}}{\alpha}\exp(\frac{pC^*\bar\nu}{p'})\|c_0\|_{L^{p'}(\Omega)}\right)},
$$

we obtain (3.19).

Now we prove (3.20). By (3.50) and Lemma 3.1, we obtain

$$
\begin{aligned}
&\|u_\varepsilon\|_{L^{N',\infty}(\Omega)} \le C(N,p)\left[M^* + |\Omega|^{\frac{1}{p^*}}(L^*)^{\frac{1}{p'}}\right] \\
&= C(N,p)\left\{2\frac{p'}{\alpha}\left[\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} + M_0\right]\right. \\
&\left.+|\Omega|^{\frac{1}{p^*}}\left[2\frac{3^{p'/p}}{\alpha^{\frac{p'}{p}+1}}(\sigma^p\|c_0\|^{p'}_{L^{p'}(\Omega)} + \|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N})\right]^{1/p'}\right\}.
\end{aligned}
$$

By a calculation similar to that made for estimate (3.51) of $|\nabla u_\varepsilon|^{p-1}$, we obtain (3.20) and therefore the conclusion of the proof of Theorem 3.2 under the assumption 1) of Theorem 2.10.

We now prove Theorem 3.2 when assumption 2) in Theorem 2.10 is satisfied, i.e. $\gamma = p-1$, $\lambda < p-1$ and $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \le r < \infty$.

Observe that the assumptions on γ and c_0 are exactly the same of the previous case, while we now assume that $\lambda < p-1$ (and not more $\lambda = p-1$ and $\|b_0\|_{L^{N,1}(\Omega)}$ small). Therefore the estimates of the various integrals in (3.23) and in (3.40) are exactly the same, except the estimate (3.26) of $\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\psi(u_\varepsilon)\right|$ and (3.44) of $\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)T_k(u_\varepsilon)\right|$.

Let us explain the method to replace the estimate (3.26) in this case. By a calculation similar to (3.26), since $\lambda < p-1$, we obtain

$$\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\psi(u_\varepsilon)\right| \le \frac{1}{A^{p'}}\left[\|1\|_{L^{\theta,\infty}(\Omega)}\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)}^{\frac{\lambda}{p-1}} + \|b_1\|_{L^1(\Omega)}\right] \tag{3.52}$$

where θ is defined by $\frac{1}{N} + \frac{\lambda}{(p-1)N'} + \frac{1}{\theta} = 1$ and where

$$A = 1 + \frac{p'}{\alpha}\left\{\|1\|_{L^{\theta,\infty}(\Omega)}\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)}^{\frac{\lambda}{p-1}} + \frac{3^{p'/p}}{p'\alpha^{p'/p}}(\|c_1\|_{L^{p'}(\Omega)}^{p'} + \|g+F\|_{(L^{p'}(\Omega))^N}^{p'}) + M_0\right\}$$

Let us explain the method to replace the estimate (3.44).
Since $\lambda < p-1$, we obtain

$$\left|\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)T_k(u_\varepsilon)\right| \le k\left[\|1\|_{L^{\theta,\infty}(\Omega)}\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)}^{\frac{\lambda}{p-1}} + \|b_1\|_{L^1(\Omega)}\right]. \tag{3.53}$$

By replacing estimate (3.26) by (3.52) and the estimate (3.44) by (3.53), we argue as in the previous case and we obtain

$$\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \le C^{**} + \frac{p'}{\alpha}C(N,p)\left(2 + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}}\frac{3^{p'}}{\alpha}\exp(\frac{pC^*\bar\nu}{p'})\|c_0\|_{L^{p'}(\Omega)}\right)\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)}^{\frac{\lambda}{p-1}}.$$

Since $\lambda < p-1$, we obtain the estimate of $\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)}$ (without any assumption on the smallness of $\|b_0\|_{L^{N,1}(\Omega)}$). Similarly we get (3.20).

Finally let us now prove Theorem 3.2 when assumption 3) in Theorem 2.10 is satisfied, i.e. $\gamma < p-1$, $\lambda < p-1$, $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$.

Using $T_k(u_\varepsilon)$ as test function, since $\gamma < p-1$ we easily obtain

$$\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \le \tilde{M}^* k + \tilde{L}^*,$$

where

$$\tilde{M}^* = C(\alpha,\gamma,p)\left[\|1\|_{L^{\theta,\infty}(\Omega)}\|b_0\|_{L^{N,1}(\Omega)}\||\nabla u_\varepsilon|^{p-1}\|^{\frac{\lambda}{p-1}}_{L^{N',\infty}(\Omega)} + M_0\right],$$

$$\tilde{L}^* = C'(\alpha,\gamma,p)\left[\|c_0\|^{\frac{p-1}{(p-1)-\gamma}}_{L^{\frac{N}{p-1},\infty}(\Omega)} + \|c_1\|^{p'}_{L^{p'}(\Omega)} + \|g+F\|^{p'}_{(L^{p'}(\Omega))^N}\right].$$

By Lemma 3.1, we get:

$$\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \le C(N,p)\left[\tilde{M}^* + |\Omega|^{\frac{1}{N'}-\frac{1}{p'}}(\tilde{L}^*)^{\frac{1}{p'}}\right],$$

$$\|u_\varepsilon\|_{L^{N',\infty}(\Omega)} \le C(N,p)\left[\tilde{M}^* + |\Omega|^{\frac{1}{p^*}}(\tilde{L}^*)^{\frac{1}{p'}}\right],$$

and by definition of $\tilde{M}^*$, since $\lambda < p-1$ we obtain (3.19) and (3.20). ■

To conclude the proof of Theorem 2.10, we have to pass to the limit in the approximate problem (3.17). As in [BMMP3], the idea is to prove that u_ε satisfies

$$\begin{cases} -\mathrm{div}(a(x,u_\varepsilon,\nabla u_\varepsilon) + K_\varepsilon(x,u_\varepsilon)) = f_\varepsilon - \Psi_\varepsilon - \mathrm{div}(g) + \mathrm{div}(F) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus \quad \text{in } \mathcal{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{3.54}$$

where

$$\Psi_\varepsilon = H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) + G_\varepsilon(x,u_\varepsilon) \to H(x,u,\nabla u) + G(x,u) \quad \text{in } L^1(\Omega) \text{ strongly}, \tag{3.55}$$

and to apply a stability result which allows to pass to the limit in (3.55).

We devote next Section to prove the stability result, which we need.

4 A generalization of the stability results of [DMOP]

In the present Section we prove a generalization of the stability result given in [DMOP] (see also [MP] and [M]).

The main feature of our result is due to the term $-\mathrm{div}(K_\varepsilon(x,u))$, where $K_\varepsilon(x,u)$ belongs to $(L^{N',\infty}(\Omega))^N$. Since $p < N$, we have $(L^{p'}(\Omega))^N \subset (L^{N',\infty}(\Omega))^N$, which implies that $K_\varepsilon(x,u)$ does not in general belong to $(L^{p'}(\Omega))^N$ and therefore the term $-\mathrm{div}(K_\varepsilon(x,u))$ is not in general an element of the dual space $W^{-1,p'}(\Omega)$.

We explicitly remark that the proof of stability result in [DMOP] (Theorem 3.4) can be adapted to our case, however here we use a slightly different method.

In the present Section we consider a nonlinear elliptic problem which can be formally written as

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u) + K_\varepsilon(x,u)) = \mu_\varepsilon & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

where ε belongs to a sequence of positive numbers that converges to zero and the function $a : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \longmapsto \mathrm{I\!R}^N$ is a Carathéodory function which satisfies assumptions (2.9), (2.10) and (2.11)

Moreover $K_\varepsilon : \Omega \times \mathrm{I\!R} \mapsto \mathrm{I\!R}^N$ is a Carathéodory function which satisfies the growth condition (2.12), i.e.

$$|K_\varepsilon(x,s)| \le c_0(x)|s|^\gamma + c_1(x), \tag{4.2}$$

for almost every $x \in \Omega$ and for every $s \in \mathrm{I\!R}$, where c_0 and c_1 satisfy the assumptions in (2.12). Denoted by $K : \Omega \times \mathrm{I\!R} \mapsto \mathrm{I\!R}^N$ a Carathéodory function, we also assume that

$$\begin{cases} K_\varepsilon(x,s_\varepsilon) \longrightarrow K(x,s) \\ \text{ for every sequence } s_\varepsilon \in \mathrm{I\!R} \text{ such that} \\ s_\varepsilon \longrightarrow s \text{ almost everywhere in } \Omega. \end{cases} \tag{4.3}$$

Finally we assume that μ_ε is a measure in $M_b(\Omega)$ which can be decomposed as

$$\mu_\varepsilon = f_\varepsilon - \mathrm{div}(g_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus,$$

$$\mu = f - \mathrm{div}(g) + \mu_s^+ - \mu_s^-,$$

where, denoted by μ the measure in $M_b(\Omega)$ given by (2.17), we assume that

$$\begin{cases} f \in L^1(\Omega) \\ \mu_s = \mu_s^+ - \mu_s^- \text{ is a measure in } M_s(\Omega) \text{ with} \\ \text{positive and negative parts } \mu_s^+ \text{ and } \mu_s^- \text{ respectively;} \end{cases} \tag{4.4}$$

$$\begin{cases} f_\varepsilon \text{ is a sequence of functions in } L^1(\Omega) \text{ such that} \\ f_\varepsilon \longrightarrow f \quad \text{in } L^1(\Omega) \text{ weakly;} \end{cases} \tag{4.5}$$

$$\begin{cases} g_\varepsilon \text{ is a sequence of functions in } (L^{p'}(\Omega))^N \text{ such that} \\ g_\varepsilon \longrightarrow g \quad \text{in } (L^{p'}(\Omega))^N \text{ strongly;} \end{cases} \tag{4.6}$$

$$\begin{cases} \lambda_\varepsilon^\oplus \text{ is a non negative measure in } M_b(\Omega) \text{ such that} \\ \lambda_\varepsilon^\oplus \longrightarrow \mu_s^+ \text{ in the narrow topology;} \end{cases} \tag{4.7}$$

$$\begin{cases} \lambda_\varepsilon^\ominus \text{ is a non negative measure in } M_b(\Omega) \text{ such that} \\ \lambda_\varepsilon^\ominus \longrightarrow \mu_s^- \text{ in the narrow topology.} \end{cases} \tag{4.8}$$

Observe that, according to Proposition 2.5, we can decompose $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ in the following way

$$\lambda_\varepsilon^\oplus = \lambda_{\varepsilon,0}^\oplus + \lambda_{\varepsilon,s}^\oplus, \quad \lambda_\varepsilon^\ominus = \lambda_{\varepsilon,0}^\ominus + \lambda_{\varepsilon,s}^\ominus$$

with $\lambda_{\varepsilon,0}^\oplus$, $\lambda_{\varepsilon,0}^\ominus \in M_0(\Omega)$, $\lambda_{\varepsilon,0}^\oplus$, $\lambda_{\varepsilon,0}^\ominus \geq 0$ and $\lambda_{\varepsilon,s}^\oplus$, $\lambda_{\varepsilon,s}^\ominus \in M_s(\Omega)$, $\lambda_{\varepsilon,s}^\oplus$, $\lambda_{\varepsilon,s}^\ominus \geq 0$.

On the other hand, using Propositions 2.4 and 2.5, the measure μ_ε can be decomposed as

$$\begin{aligned}\mu_\varepsilon &= \mu_{\varepsilon,0} + \mu_{\varepsilon,s} \\ &= \mu_{\varepsilon,0} + \mu_{\varepsilon,s}^+ - \mu_{\varepsilon,s}^-,\end{aligned}$$

where $\mu_{\varepsilon,0}$ is a measure in $M_0(\Omega)$ and where $\mu_{\varepsilon,s}^+$ and $\mu_{\varepsilon,s}^-$ (the positive and the negative parts of $\mu_{\varepsilon,s}$) are two nonnegative measures in $M_s(\Omega)$, which are concentrated on two disjoint subsets E_ε^+ and E_ε^- of zero p-capacity.

Therefore we can conclude, by the definition of μ_ε, that

$$\mu_{\varepsilon,0} = f_\varepsilon + \mathrm{div}(g_\varepsilon) + \lambda_{\varepsilon,0}^\oplus - \lambda_{\varepsilon,0}^\ominus, \tag{4.9}$$

and

$$\mu_{\varepsilon,s} = \lambda_{\varepsilon,s}^\oplus - \lambda_{\varepsilon,s}^\ominus.$$

We have (cf. [DMOP], Remark 3.5)

$$0 \leq \mu_{\varepsilon,s}^+ \leq \lambda_{\varepsilon,s}^\oplus \quad 0 \leq \mu_{\varepsilon,s}^- \leq \lambda_{\varepsilon,s}^\ominus. \tag{4.10}$$

We prove the following theorem

Theorem 4.1 *Assume that (2.9), (2.10), (2.11), (4.2), (4.3), (4.4)–(4.8) hold true with* $\gamma = p-1$ *and* $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r < +\infty$. *Let* u_ε *be a renormalized solution of (4.1). Then, up to a subsequence still indexed by* ε, u_ε *converges almost everywhere to a renormalized solution* u *to the problem*

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u) + K(x,u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

and moreover $T_k(u_\varepsilon) \to T_k(u)$ *in* $W_0^{1,p}(\Omega)$ *strongly,* $\forall k > 0$.

The same conclusions hold true if we assume that (4.2) holds true with $\gamma < p-1$ *and* $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$.

Remark 4.2 The stability result given by Theorem 4.1 coincides with the stability result proved in [DMOP] (Theorem 3.4) (see also [M]) when $K_\varepsilon = 0$. Therefore we prove an extension of such a result. We explicitly remark that our proof is slightly different. As in [DMOP], our stability result implies an existence result for renormalized solution to the problem (4.11); such a result extends the existence result proved in [BGu2] in the case where μ belongs to $L^1(\Omega)$.

Remark 4.3 Observe that, by the growth assumption (4.2) and the convergence assumption (4.3) on K_ε, we deduce

$$|K(x,s)| \leq c_0(x)|s|^\gamma + c_1(x), \tag{4.12}$$

for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$.

Observe also that, since a renormalized solution u_ε to problem (4.1) and a renormalized solution u to the problem (4.11) satisfy the conditions

$$|u_\varepsilon|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega) \quad \text{and} \quad |u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega),$$

by growth assumptions (4.2) on K_ε and growth condition (4.12) on K, we deduce that

$$|K_\varepsilon(x,u_\varepsilon)| \in L^{\frac{N}{N-1},r}(\Omega) \quad \text{and} \quad |K(x,u)| \in L^{\frac{N}{N-1},r}(\Omega), \qquad \frac{N}{p-1} \leq r \leq +\infty.$$

On the other hand, since $p < N$, $L^{p'}(\Omega) \subset L^{\frac{N}{N-1},\infty}(\Omega)$. This implies that, in general, $K_\varepsilon(x,u_\varepsilon)$ and $K(x,u)$ does not belong to $(L^{p'}(\Omega))^N$ and therefore the terms $-\mathrm{div}(K_\varepsilon(x,u_\varepsilon))$ and $-\mathrm{div}(K(x,u))$ are in general not elements of the dual space $W^{-1,p'}(\Omega)$.

Remark 4.4 Observe that Theorem 4.1 holds true under the same assumption, if we replace the right-hand side by a more general datum $\mu - \mathrm{div}(F)$, with $F \in (L^{p'}(\Omega))^N$. Indeed $K_\varepsilon(x,s)$ (resp. $K(x,s)$) can be replaced by $K_\varepsilon(x,s) - F(x)$ (resp. $K(x,s) - F$) which verifies conditions (4.2) and (4.3) (with c_1 replaced by $c_1 + |F|$).

Proof of Theorem 4.1

We begin by proving Theorem 4.1 under the assumptions that $\gamma = p-1$, $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \leq r < \infty$. Observe that from now on, such assumptions will be used only to obtain a priori estimates for $|u_\varepsilon|^{p-1}$ and $|\nabla u_\varepsilon|^{p-1}$ in the preliminary step below and to prove (4.24) and (4.25) in the second step below.

Preliminary step. We begin by proving the following a priori estimate for the renormalized solution u_ε

$$\||u_\varepsilon|^{p-1}\|_{L^{\frac{N}{p-1},\infty}(\Omega)} \leq c, \tag{4.13}$$

$$\||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} \leq c, \tag{4.14}$$

where c is a positive constant depending only on the data, which does not depend on ε. This is done by adapting to the case of renormalized solution u_ε the techniques used in the proof of Theorem 3.2 above which allows us to prove the same a priori estimate for the weak solution of problem (3.17). Since the arguments are very similar to that of Theorem 3.2, we do not give the details, but we only give a sketch of the proof.

Since u_ε is a renormalized solution, we have

$$
\begin{aligned}
&\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\, h'(u_\varepsilon)v+\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v\, h(u_\varepsilon)\\
&\quad+\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\, h'(u_\varepsilon)v+\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla v\, h(u_\varepsilon)\\
&=\int_\Omega f_\varepsilon h(u_\varepsilon)v+\int_\Omega g_\varepsilon\cdot\nabla u_\varepsilon\, h'(u_\varepsilon)v+\int_\Omega g_\varepsilon\cdot\nabla v\, h(u_\varepsilon)\\
&\quad+\int_\Omega \lambda^{\oplus}_{\varepsilon,0}h(u_\varepsilon)v+\int_\Omega \lambda^{\ominus}_{\varepsilon,0}h(u_\varepsilon)v,
\end{aligned}
\tag{4.15}
$$

for every $v\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, for all $h\in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $h(u_\varepsilon)v\in W_0^{1,p}(\Omega)$.

Firstly we use in (4.15) the test function $h_n(u_\varepsilon)\psi(T_{2n}(u_\varepsilon))$ where h_n is defined by (2.29) and ψ is defined by (3.21) (with $A\equiv 1$). Then by calculations similar to that of first step of the proof of Theorem 3.2, we get the following estimate for the level sets of u_ε,

$$
\operatorname{meas}\{|u_\varepsilon|\ge \exp(C^*\eta)\}\le \frac{1}{\eta^p},\quad \forall\eta>0, \tag{4.16}
$$

where C^* is defined by (3.38).

Secondly we use in (4.15) the test function $h_n(u_\varepsilon)T_k(u_\varepsilon)$ for every $k>0$, where $h_n(s)$ is defined by (2.29) and, using calculations similar to that of second step of the proof of Theorem 3.2, we get

$$
\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N}\le \tilde{M}k+\tilde{L},\quad \forall k>0 \tag{4.17}
$$

for some $\tilde{M}>0$ and $\tilde{L}>0$. This implies, by Lemma 3.1, (4.13) and (4.14).

Estimate (4.17) and the growth assumptions on K_ε, since the operator a is strictly monotone, allow us to use standard techniques (see, e.g., [BMu] and [DMOP]) to say that there exists a measurable function $u:\Omega\longmapsto\bar{\mathbb{R}}$, finite almost everywhere in Ω and such that, up to a subsequence still indexed by ε,

$$
T_k(u_\varepsilon)\longrightarrow T_k(u)\quad \text{in } W_0^{1,p}(\Omega) \text{ weakly, } \forall k>0, \tag{4.18}
$$

$$
u_\varepsilon\longrightarrow u\quad \text{almost everywhere in } \Omega, \tag{4.19}
$$

$$
\nabla u_\varepsilon\longrightarrow \nabla u\quad \text{almost everywhere in } \Omega, \tag{4.20}
$$

as ε goes to 0.

First step. In this step we prove that the function u is solution of (4.11) in the sense of distributions.

By assumption (4.3) and (4.19), it follows that $K_\varepsilon(x,u_\varepsilon)$ converges to $K(x,u)$ almost everywhere in Ω. Moreover, the growth assumption (4.2) on K_ε and the estimate

(4.13) of $|u_\varepsilon|^{p-1}$ imply that $|K_\varepsilon(x,u_\varepsilon)|$ is bounded in $L^{N',\infty}(\Omega)$. Therefore Lebesgue convergence theorem gives

$$K_\varepsilon(x,u_\varepsilon) \longrightarrow K(x,u) \quad \text{in } L^r(\Omega) \text{ strongly, } \forall r < N/(N-1). \tag{4.21}$$

In a similar way by (2.10) on a, (4.13), (4.14), (4.19) and (4.20) and Lebesgue convergence theorem, we get

$$a(x,u_\varepsilon,\nabla u_\varepsilon) \longrightarrow a(x,u,\nabla u) \quad \text{in } L^r(\Omega) \text{ strongly } \forall r < N/(N-1). \tag{4.22}$$

Since u_ε is a renormalized solution, it is also a solution in the sense of distribution of (4.1), that is

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\phi + \int_\Omega K(x,u_\varepsilon)\cdot\nabla\phi = \int_\Omega \phi\, d\mu_\varepsilon, \tag{4.23}$$

for all $\phi \in C_0^\infty(\Omega)$, (cf. Remark 2.9 above). Therefore, using the convergences in (4.21) and (4.22) and the assumptions (4.4) - (4.8), we easily can pass to the limit in (4.23) and obtain that u is solution in the sense of distribution of (4.11).

Second step. In this step we prove that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{n<|u_\varepsilon|<2n\}}|K_\varepsilon(x,u_\varepsilon)||\nabla u_\varepsilon| = 0, \tag{4.24}$$

and

$$\limsup_{n\to\infty}\frac{1}{n}\int_{\{n<|u|<2n\}}|K(x,u)||\nabla u| = 0. \tag{4.25}$$

Since by the generalized Sobolev inequality (2.7), $T_k(u_\varepsilon) \in L^{p^*,t}(\Omega)$ for any $p \le t \le +\infty$, by the growth condition (3.10) on K_ε, the generalized Hölder inequality (2.5) and (4.17), we get

$$\begin{aligned}
\frac{1}{n}\int_{n<|u_\varepsilon|<2n}|K_\varepsilon(x,u_\varepsilon)||\nabla u_\varepsilon| &\le \frac{1}{n}\int_{n<|u_\varepsilon|<2n} c_0|u_\varepsilon|^{p-1}|\nabla u_\varepsilon| + \frac{1}{n}\int_{n<|u_\varepsilon|<2n} c_1|\nabla u_\varepsilon| \\
&\le \frac{1}{n}\int_{n<|u_\varepsilon|<2n} c_0|T_{2n}(u_\varepsilon)|^{p-1}|\nabla T_{2n}(u_\varepsilon)| + \frac{1}{n}\int_{n<|u_\varepsilon|<2n} c_1|\nabla T_{2n}(u_\varepsilon)| \\
&\le \frac{c}{n}\|\nabla T_{2n}(u_\varepsilon)\|^p_{(L^p(\Omega))^N}\|c_0\|_{L^{\frac{N}{p-1},r}(\{n<|u_\varepsilon|<2n\})} \\
&\quad + \frac{1}{n^{1/p'}}\|c_1\|_{L^{p'}(\Omega)}\frac{1}{n^{1/p}}\|\nabla T_{2n}(u_\varepsilon)\|_{(L^p(\Omega))^N} \\
&\le c\|c_0\|_{L^{\frac{N}{p-1},r}(\{n<|u_\varepsilon|<2n\})} + c\frac{1}{n^{1/p'}}\|c_1\|_{L^{p'}(\Omega)}.
\end{aligned} \tag{4.26}$$

Moreover

$$\|c_0\|_{L^{\frac{N}{p-1},r}(\{n<|u_\varepsilon|<2n\})} = \left(\int_0^{|\{n<|u_\varepsilon|<2n\}|}[(c_0)^*(t)t^{\frac{p-1}{N}}]^r\frac{dt}{t}\right)^{1/r}$$

tends to 0 when firstly ε goes to zero and then n goes to infinity (indeed u_ε converges to u a.e. in Ω and u is finite a.e. in Ω). Therefore we obtain (4.24) by using Fatou Lemma in (4.26).

In a similar way, we get also (4.25).

Third step. In this step we prove a slightly different version of Lemma 6.1 of [DMOP], i.e. we prove that

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\varphi \le \int_\Omega \varphi d\mu_s^+ \tag{4.27}$$

and

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{-2n<u_\varepsilon<-n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\varphi \le \int_\Omega \varphi d\mu_s^-, \tag{4.28}$$

for any $\varphi\in C^1(\bar\Omega)$ with $\varphi\ge 0$.

We begin by proving (4.27).

For any $n\ge 1$, let us define the $s_n:\mathbb{R}\longmapsto\mathbb{R}$ and $h_\eta:\mathbb{R}\longmapsto\mathbb{R}$ by

$$s_n(r)=\frac{T_{2n}(r)-T_n(r)}{n},$$

$$h_\eta(r)=1-|s_\eta(r)|.$$

Denote $v^+(x)=\max\{0,v(x)\}$ and $v^-(x)=\max\{0,-v(x)\}$ for almost every $x\in\Omega$. Using the test function $h_\eta(u_\varepsilon)s_n(u_\varepsilon^+)\varphi$ in (4.15) and letting η goes to infinity, we have

$$\begin{aligned}
&\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi s_n(u_\varepsilon^+)+\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\varphi\\
&+\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla\varphi s_n(u_\varepsilon^+)+\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\varphi\\
&=\int_\Omega f_\varepsilon s_n(u_\varepsilon^+)\varphi+\int_\Omega g_\varepsilon\cdot\nabla\varphi s_n(u_\varepsilon^+)+\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} g_\varepsilon\cdot\nabla u_\varepsilon\varphi\\
&\quad+\int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^{\oplus}+\int_\Omega\varphi d\mu_{\varepsilon,s}^+-\int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^{\ominus},
\end{aligned}\tag{4.29}$$

for any $\varphi\in C^1(\bar\Omega)$ nonnegative.

Now we pass to the limit in the various terms in (4.29), first as $\varepsilon\to 0$ and then as $n\to\infty$.

Since $s_n(u_\varepsilon^+)$ is bounded by 1, we have

$$s_n(u_\varepsilon^+)\to s_n(u^+)\text{ almost everywhere and weakly-}*\text{ in }L^\infty(\Omega). \tag{4.30}$$

Therefore, from (4.21) and (4.22), it follows that

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi s_n(u_\varepsilon^+)=\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi s_n(u^+),$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla\varphi s_n(u_\varepsilon^+)=\int_\Omega K(x,u)\cdot\nabla\varphi s_n(u^+).$$

Since both the functions $a(x,u,\nabla u)$ and $K(x,u)$ belong to $(L^r(\Omega))^N$ for $1\le r<N/(N-1)$, $s_n(u^+)$ is bounded by 1, $s_n(u^+)\to 0$ almost everywhere in Ω and $\varphi\in C^1(\bar\Omega)$, Lebesgue convergence theorem implies that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi s_n(u_\varepsilon^+)=\lim_{n\to+\infty}\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi s_n(u^+)=0, \tag{4.31}$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega K_\varepsilon(x,u_\varepsilon)\cdot\nabla\varphi s_n(u_\varepsilon^+)=\lim_{n\to+\infty}\int_\Omega s_n(u^+)K(x,u)\cdot\nabla\varphi=0. \tag{4.32}$$

Moreover, from (4.24), we get

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{n<|u_\varepsilon|<2n\}}|K_\varepsilon(x,u_\varepsilon)||\nabla u_\varepsilon|\varphi=0. \tag{4.33}$$

Since f_ε converges weakly to f in $L^1(\Omega)$ and since $s_n(u_\varepsilon^+)$ converges to $s_n(u^+)$ almost everywhere while $|s_n(u_\varepsilon^+)|\le 1$ a.e., Proposition 2.6 gives

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon s_n(u_\varepsilon^+)\varphi=\lim_{n\to\infty}\int_\Omega f s_n(u^+)\varphi=0. \tag{4.34}$$

Furthermore, by (4.6), since $s_n(u_\varepsilon^+)$ converges to $s_n(u^+)$ almost everywhere, we get

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega g_\varepsilon\cdot\nabla\varphi s_n(u_\varepsilon^+)=\lim_{n\to\infty}\int_\Omega g\cdot\nabla\varphi s_n(u^+)=0. \tag{4.35}$$

On the other hand, Hölder's inequality implies

$$\frac{1}{n}\left|\int_{\{n<u_\varepsilon<2n\}}g_\varepsilon\cdot\nabla u_\varepsilon\varphi\right|\le\|\varphi\|_{L^\infty(\Omega)}\frac{1}{n^{1/p'}}\|g_\varepsilon\|_{(L^{p'}(\Omega))^N}\left(\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}}|\nabla u_\varepsilon|^p\right)^{1/p},$$

from which it follows (using (4.17))

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}}g_\varepsilon\cdot\nabla u_\varepsilon\varphi=0. \tag{4.36}$$

Since $0\le\mu_{\varepsilon,s}^+\le\lambda_{\varepsilon,s}^\oplus$ (see (4.10)), $\varphi\ge 0$, $0\le s_n(u_\varepsilon^+)\le 1$ and since $\lambda_{\varepsilon,0}^\ominus$ is a positive measure, we have

$$\begin{aligned}\int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^\oplus+\int_\Omega\varphi d\mu_{\varepsilon,s}^+&\le\int_\Omega\varphi d\lambda_{\varepsilon,0}^\oplus+\int_\Omega\varphi d\lambda_{\varepsilon,s}^\oplus-\int_\Omega s_n(u_\varepsilon^+)\varphi d\lambda_{\varepsilon,0}^\ominus\\&\le\int_\Omega\varphi d\lambda_\varepsilon^\oplus.\end{aligned} \tag{4.37}$$

Combining (4.29)–(4.37), we get for every $\varphi \in C^1(\bar{\Omega})$ with $\varphi \geq 0$

$$\frac{1}{n}\int_{\{n<u_\varepsilon<2n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\varphi \leq \omega(\varepsilon,n)+\int_\Omega \varphi d\lambda_\varepsilon^\oplus,$$

where $\omega(\varepsilon,n)$ denotes a function such that $\lim\limits_{n\to\infty}\limsup\limits_{\varepsilon\to 0}\omega(\varepsilon,n)=0$.

Since $\lambda_\varepsilon^\oplus \to \mu_s^+$ in the narrow topology, we obtain (4.27).

Using $s_n(u_\varepsilon^-)\varphi$ and similar arguments we have also (4.28).

Fourth step. In this step we prove that the limit function u is a renormalized solution to the problem (4.11).

We begin by proving that for every $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $h(u)v \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u)\cdot\nabla u\, h'(u)v+\int_\Omega a(x,u,\nabla u)\cdot\nabla v\, h(u)\\&\quad+\int_\Omega K(x,u)\cdot\nabla u\, h'(u)v+\int_\Omega K(x,u)\cdot\nabla v\, h(u)\\&=\int_\Omega fh(u)v+\int_\Omega g\cdot\nabla u\, h'(u)v+\int_\Omega g\cdot\nabla v\, h(u).\end{aligned}\tag{4.38}$$

For any $\delta > 0$, let us consider two cut-off functions ψ_δ^+ and ψ_δ^- belonging to $C_0^\infty(\Omega)$ such that

$$0\leq\psi_\delta^+\leq 1,\quad 0\leq\psi_\delta^-\leq 1,\quad \mathrm{supp}(\psi_\delta^+)\cap\mathrm{supp}(\psi_\delta^-)=\emptyset;\tag{4.39}$$

$$\lim_{\delta\to 0}\int_\Omega|\nabla\psi_\delta^+|^p=\lim_{\delta\to 0}\int_\Omega|\nabla\psi_\delta^-|^p=0;\tag{4.40}$$

$$\lim_{\delta\to 0}\int_\Omega\psi_\delta^+ d\mu_s^-=0,\quad \lim_{\delta\to 0}\int_\Omega\psi_\delta^- d\mu_s^+=0;\tag{4.41}$$

$$\lim_{\delta\to 0}\int_\Omega(1-\psi_\delta^+)d\mu_s^+=0,\quad \lim_{\delta\to 0}\int_\Omega(1-\psi_\delta^-)d\mu_s^-=0;\tag{4.42}$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_\Omega\psi_\delta^- d\lambda_\varepsilon^\oplus=0,\quad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_\Omega\psi_\delta^+ d\lambda_\varepsilon^\ominus=0;\tag{4.43}$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_\Omega(1-\psi_\delta^+)d\lambda_\varepsilon^\oplus=0,\quad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_\Omega(1-\psi_\delta^-)d\lambda_\varepsilon^\ominus=0.\tag{4.44}$$

The existence of such cut-off functions is proved in Lemma 5.1 of [DMOP].

Moreover let h be an element of $W^{1,\infty}(\mathbb{R})$ with compact support and let $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $h(u)v$ belongs to $W_0^{1,p}(\Omega)$.

Using the test function $h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)$ in (4.15), where h_n is defined by (2.29), we get

$$\begin{aligned}
&\int_\Omega h'_\varepsilon(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla u_\varepsilon\\
&+\int_\Omega h'(u)h_n(u_\varepsilon)v(1-\psi_\delta^+-\psi_\delta^-)[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla u\\
&+\int_\Omega h(u)h_n(u_\varepsilon)(1-\psi_\delta^+-\psi_\delta^-)[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla v\\
&+\int_\Omega h(u)h_n(u_\varepsilon)v[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla(1-\psi_\delta^+-\psi_\delta^-)\\
&=\int_\Omega f_\varepsilon h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)+\int_\Omega g_\varepsilon\nabla[h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)]\\
&+\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^\oplus-\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^\ominus.
\end{aligned}\tag{4.45}$$

We now pass to the limit in (4.45) first as $\varepsilon\to0$, then as $n\to\infty$ and finally as $\delta\to0$.

From (4.24), we have

$$\lim_{n\to\infty}\limsup_{\varepsilon\to0}\left|\int_\Omega h'_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)K_\varepsilon(x,u_\varepsilon)\cdot\nabla u_\varepsilon\right|=0.\tag{4.46}$$

By (4.27), since $(1-\psi_\delta^+-\psi_\delta^-)a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon$ is positive, one has

$$\begin{aligned}
0\le&\limsup_{n\to\infty}\limsup_{\varepsilon\to0}\frac{1}{n}\int_{\{n<|u_\varepsilon|<2n\}}|h(u)v|(1-\psi_\delta^+-\psi_\delta^-)a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\\
&\le\|h\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\Omega)}\left(\int_\Omega(1-\psi_\delta^+-\psi_\delta^-)d\mu_s^++\int_\Omega(1-\psi_\delta^+-\psi_\delta^-)d\mu_s^-\right),
\end{aligned}\tag{4.47}$$

and therefore (4.41) and (4.42) lead to

$$\lim_{\delta\to0}\limsup_{n\to\infty}\limsup_{\varepsilon\to0}\frac{1}{n}\int_{\{n<|u_\varepsilon|<2n\}}|h(u)v|(1-\psi_\delta^+-\psi_\delta^-)a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon=0.\tag{4.48}$$

By the weak convergence of $T_{2n}(u_\varepsilon)$ in $W_0^{1,p}(\Omega)$, we deduce that $a(x,T_{2n}(u_\varepsilon),\nabla T_{2n}(u_\varepsilon))$ converges to $a(x,T_{2n}(u),\nabla T_{2n}(u))$ weakly in $(L^p(\Omega))^N$. Moreover, since $h_n(u_\varepsilon)$ is bounded by 1 and converges to $h_n(u)$ a.e., this implies that

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega h'(u)h_n(u_\varepsilon)v(1-\psi_\delta^+-\psi_\delta^-)[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla u\\
&\quad=\int_\Omega h'(u)h_n(u)v(1-\psi_\delta^+-\psi_\delta^-)[a(x,u,\nabla u)+K(x,u)]\cdot\nabla u
\end{aligned}$$

On the other hand supp(h) is compact, that is supp$(h)\subset]-M,M[$ for some $M>0$. Therefore $a(x,u,\nabla u)$ in the last integral can be replaced by $a(x,T_M(u),\nabla T_M(u))$ which belongs to $(L^{p'}(\Omega))^N$. Moreover, by (4.40), ψ_δ^+ and ψ_δ^- converge to 0 strongly in $W^{1,p}(\Omega)$ as δ tends to zero. Thus, we obtain that

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega h'(u)h_n(u_\varepsilon)v(1-\psi_\delta^+-\psi_\delta^-)[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla u\\&\qquad=\int_\Omega h'(u)v[a(x,u,\nabla u)+K(x,u)]\cdot\nabla u.\end{aligned}\tag{4.49}$$

Similar arguments yield that

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega h(u)h_n(u_\varepsilon)(1-\psi_\delta^+-\psi_\delta^-)[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla v\\&=\int_\Omega h(u)[a(x,u,\nabla u)+K(x,u)]\cdot\nabla v\end{aligned}\tag{4.50}$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega h(u)h_n(u_\varepsilon)v[a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)]\cdot\nabla(1-\psi_\delta^+-\psi_\delta^-)=0.\tag{4.51}$$

Using Proposition 2.6, the point-wise convergence of u_ε (4.19), the definition of h_n (2.29) and the property (4.40) of ψ_δ^+ and ψ_δ^-, we obtain that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)=\int_\Omega fh(u)v.\tag{4.52}$$

Similarly it can be shown that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega g_\varepsilon\nabla[h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)]=\int_\Omega g\cdot\nabla[h(u)v].\tag{4.53}$$

Since $\lambda_{\varepsilon,0}^\oplus$ is a nonnegative measure, we have

$$\left|\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^\oplus\right|\le\|h\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\Omega)}\int_\Omega(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^\oplus.$$

Therefore the inequality $0\le\lambda_{\varepsilon,0}^\oplus\le\lambda_\varepsilon^\oplus$, and (4.43) and (4.44), imply

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^\oplus\right|=0.\tag{4.54}$$

In the same way we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega h_n(u_\varepsilon)h(u)v(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\varepsilon,0}^\ominus\right|=0.$$

This conclude the proof of (4.38).

Since we have proved that u satisfies (4.25), to conclude the proof that u is a renormalized solution of (4.11), it remains to prove that we have

$$\lim_{n\to+\infty}\frac{1}{n}\int_{n<u<2n} a(x,u,\nabla u)\cdot\nabla u\,\varphi=\int_\Omega \varphi d\mu_s^+, \tag{4.55}$$

$$\lim_{n\to+\infty}\frac{1}{n}\int_{-2n<u<-n} a(x,u,\nabla u)\cdot\nabla u\,\varphi=\int_\Omega \varphi d\mu_s^-. \tag{4.56}$$

for any φ belonging to $C_b^0(\Omega)$.

Since $a(x,u,\nabla u)\cdot\nabla u$ is positive, Remark 2.2 allows us to prove (4.55) and (4.56) for $\varphi\in C^\infty(\bar\Omega)$.

As a consequence of (4.27), the convergence almost everywhere of u_ε to u (4.19) and Fatou lemma, we have

$$\limsup_{n\to+\infty}\frac{1}{n}\int_{n<u<2n} a(x,u,\nabla u)\cdot\nabla u\,\varphi\le\int_\Omega \varphi d\mu_s^+, \tag{4.57}$$

$$\limsup_{n\to+\infty}\frac{1}{n}\int_{-2n<u<-n} a(x,u,\nabla u)\cdot\nabla u\,\varphi\le\int_\Omega \varphi d\mu_s^-, \tag{4.58}$$

for any $\varphi\in C^1(\bar\Omega)$ with $\varphi\ge 0$.

On the other hand, since u is solution of (4.11) in the sense of distributions, we can use $h_n(u)\psi$ as test function in (4.38), where h_n is defined by (2.29) and $\psi\in C_0^\infty(\Omega)$. Then, letting n goes to $+\infty$, we obtain

$$\lim_{n\to\infty}\int_\Omega h_n'(u)a(x,u,\nabla u)\cdot\nabla u\psi=-\int_\Omega\psi d\mu_s^+ +\int_\Omega \psi d\mu_s^-, \tag{4.59}$$

for any $\psi\in C_0^\infty(\Omega)$.

Let $\varphi\in C^1(\bar\Omega)$ with $\varphi\ge 0$. Since $0\le\psi_\delta^+\le 1$, $0\le\psi_\delta^-\le 1$, we have $0\le\varphi(1-\psi_\delta^-)\psi_\delta^+\le\varphi$ and since $\psi_\delta^+,\psi_\delta^-\in C_0^\infty(\Omega)$, we get $\varphi(1-\psi_\delta^-)\psi_\delta^+\in C_0^\infty(\Omega)$. Recalling that $h_n'(s)=\frac{1}{n}(-\chi_{\{n<s<2n\}}+\chi_{\{-2n<s<-n\}})$ a.e. in $\mathbb{R}$, from (4.59), we get

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\int_{\{n<u<2n\}} a(x,u,\nabla u)\cdot\nabla u\varphi &\ge\liminf_{n\to\infty}\frac{1}{n}\int_{\{n<u<2n\}} a(x,u,\nabla u)\cdot\nabla u\varphi(1-\psi_\delta^-)\psi_\delta^+\\ &\ge\lim_{n\to\infty}\Big(-\int_\Omega h_n'(u)a(x,u,\nabla u)\cdot\nabla u\varphi(1-\psi_\delta^-)\psi_\delta^+\Big)\\ &=\int_\Omega\varphi(1-\psi_\delta^-)\psi_\delta^+d\mu_s^+-\int_\Omega\varphi(1-\psi_\delta^-)\psi_\delta^+d\mu_s^-\\ &=\int_\Omega\varphi\psi_\delta^+d\mu_s^+-\int_\Omega\varphi\psi_\delta^-\psi_\delta^+d\mu_s^+-\int_\Omega\varphi(1-\psi_\delta^-)\psi_\delta^+d\mu_s^-.\end{aligned}$$

Now we pass to the limit, as $\delta\to 0$ in the above inequality. By (4.41) and (4.42) we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\int_{\{n<u<2n\}} a(x,u,\nabla u)\cdot\nabla u\varphi\ge\int_\Omega\varphi d\mu_s^+$$

for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi \geq 0$. The above inequality and (4.57) imply that (4.55) holds true for any $\varphi \geq 0$, $\varphi \in C^1(\bar{\Omega})$ and therefore, it holds true also for any $\varphi \in C^\infty(\bar{\Omega})$.

Similar arguments with the function $\varphi(1-\psi_\delta^+)\psi_\delta^-$ give (4.56). This conclude the proof that the function u is a renormalized solution of (4.11).

Fifth step. In this step we prove that $T_k(u_\varepsilon)$ converges to $T_k(u)$ strongly in $W_0^{1,p}(\Omega)$ as ε goes to 0, for any $k > 0$. This is obtained by standard arguments (see, e.g. [BMu], [DMOP]), so that we only sketch the proof.

On the one hand the coercivity of the operator a, the point-wise convergence of u_ε and of ∇u_ε (see (4.19) and (4.20)) together with Fatou lemma imply that for any $k > 0$

$$\int_\Omega a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) \leq \liminf_{\varepsilon\to 0} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon). \quad (4.60)$$

On the other hand, since u is a renormalized solution to (4.11), using $h_n(u_\varepsilon)T_k(u_\varepsilon)$ as test function in (4.15), letting first n goes to infinity and then ε goes to zero allow to show that

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) \leq \int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u). \quad (4.61)$$

Therefore, (4.60) and (4.61) allow to conclude that

$$\lim_{\varepsilon\to 0} \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) = \int_\Omega a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u). \quad (4.62)$$

Standard integration arguments, the coercivity assumption (2.9) on a, the strict monotonocity assumption (2.10) on a, the point-wise convergence of $DT_k(u_\varepsilon)$ and Vitali's theorem imply that

$$\nabla T_k(u_\varepsilon) \longrightarrow \nabla T_k(u) \quad \text{strongly in } (L^p(\Omega))^N.$$

This conclude the proof of fifth step and of Theorem 4.1 in the case where $\gamma = p-1$ and $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$.

Now let us prove Theorem 4.1 under the assumption that (4.2) holds true with $\gamma < p-1$ and $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$. As observed at the beginning of the proof of Theorem 4.1, the assumptions on γ and c_0 are used in the proof just to prove the estimates (4.13) on $|u_\varepsilon|^{p-1}$ and (4.14) on $|\nabla u_\varepsilon|^{p-1}$ in the preliminary step and to prove (4.24) and (4.25) in the second step. Therefore we will just prove that, under the new assumptions on γ and c_0, (i.e. $\gamma < p-1$ and $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$), (4.13), (4.14), (4.24) and (4.25) hold true. Since the rest of the proof is the same, this will conclude the proof.

In order to prove estimates (4.13) and (4.14) we follow by adapting the techniques used in the proof of Theorem 3.2 above. Firstly we use in (4.15) the test function

$h_n(u_\varepsilon)T_k(u_\varepsilon)$ for every $k > 0$, where $h_n(s)$ is defined by (2.29) and, using calculations similar to that of second step of the proof of Theorem 3.2, we get

$$\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \le \tilde{M}^{**}k + \tilde{L}^{**}, \quad \forall k > 0, \tag{4.63}$$

for some $\tilde{M}^{**} > 0$ and $\tilde{L}^{**} > 0$. This implies, by Lemma 3.1, (4.13) and (4.14).

In order to prove (4.24) we use arguments similar to the proof of (4.26). Therefore, since by generalized Sobolev inequality (2.7), $T_k(u_\varepsilon) \in L^{p^*,t}(\Omega)$ for any $p \le t \le +\infty$, by the growth condition (3.10) on K_ε, generalized Hölder inequality (2.5) and (4.17), we get

$$\frac{1}{n}\int_{n<|u_\varepsilon|<2n} |K_\varepsilon(x,u_\varepsilon)||\nabla u_\varepsilon| \le \frac{c}{n^{(\gamma/(p-1))'}}\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)} + c\frac{1}{n^{1/p'}}\|c_1\|_{L^{p'}(\Omega)}.$$

This gives (4.24). In a similar way, we get also (4.25). ■

Remark 4.5 Observe that the proof of Theorem 4.1 heavily needs conditions (4.24) and (4.25) (for example, (4.24) and (4.25) are crucial to obtain (4.27) and (4.28)). This led us to assume $\gamma = p-1$ and $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < \infty$ or $\gamma < p-1$ and $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$. We explicitly remark that such assumptions are not due to the method which we use, but the same assumptions are needed if one follows the proof of [DMOP].

Remark 4.6 Observe that we could prove Theorem 4.1 under the assumptions that $\gamma = p-1$, $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$ with the norm of $\|c_0\|_{L^{\frac{N}{p-1},\infty}(\Omega)}$ is small enough and that the right-hand side μ is a measure belonging to $M_0(\Omega)$. Indeed, using the test function $h_n(u_\varepsilon)T_k(u_\varepsilon)$ for every $k > 0$, where $h_n(s)$ is defined by (2.29) we easily get

$$\|\nabla T_k(u_\varepsilon)\|^p_{(L^p(\Omega))^N} \le \bar{M}k + \bar{L}, \quad \forall k > 0 \tag{4.64}$$

for some $\bar{M}$ and $\bar{L}$. This implies, by Lemma 3.1, (4.13) and (4.14).

Moreover, since u is a renormalized solution of (4.11), it results

$$\lim_{n\to+\infty}\frac{1}{n}\int_{n<u<2n} a(x,u,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi d\mu^+_{\varepsilon,s}, \tag{4.65}$$

$$\lim_{n\to+\infty}\frac{1}{n}\int_{-2n<u<-n} a(x,u,\nabla u)\cdot\nabla u\,\varphi = \int_\Omega \varphi d\mu^-_{\varepsilon,s}, \tag{4.66}$$

for every $\varphi \in C^0_b(\Omega)$. On the other hand μ is a measure belonging to $M_0(\Omega)$. Therefore we have $\mu^+_s = \mu^-_s = 0$ in (4.65) and (4.66) which imply

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\frac{1}{n}\int_{\{n<|u_\varepsilon|<2n\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\varphi = 0. \tag{4.67}$$

Once we have (4.64) and (4.67), we can make the same proof of Theorem 4.1.

5 Proof of Theorem 2.10

Our goal is to prove that the terms $H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$ and $G_\varepsilon(x,u_\varepsilon)$ converge strongly in $L^1(\Omega)$. This allows us to reconduce the proof to the stability result Theorem 4.1, by using arguments similar to these used in [BMMP3].

In the rest of this Subsection, c denotes a generic constant, which does not depend on ε but can vary from line to line.

We begin by proving that the solution u_ε of (3.17) satisfies

$$\begin{cases} -\mathrm{div}(a(x,u_\varepsilon,\nabla u_\varepsilon)+K_\varepsilon(x,u_\varepsilon)) = \Phi_\varepsilon - \mathrm{div}(g) + \mathrm{div}(F) & \text{in } \mathcal{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{5.1}$$

where

$$\begin{cases} \Phi_\varepsilon = f_\varepsilon - H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) - G_\varepsilon(x,u_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus, \\ \text{is bounded in } L^1(\Omega). \end{cases}$$

Indeed using the growth condition (3.12) on H_ε, Theorem 3.2 and the generalized Hölder inequality (2.5), we get

$$\begin{aligned} \|H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\|_{L^1(\Omega)} &= \int_\Omega |H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)| \\ &\le c\int_\Omega b_0|\nabla u_\varepsilon|^{p-1} + \int_\Omega b_1 \\ &\le \|b_0\|_{L^{N,1}(\Omega)} \||\nabla u_\varepsilon|^{p-1}\|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)} \le c. \end{aligned} \tag{5.2}$$

Moreover using the growth condition (3.15) on G_ε, (3.19), the generalized Hölder inequality (2.5) and the fact that $0 \le r < \frac{N(p-1)}{N-p}$, we get

$$\begin{aligned} \|G_\varepsilon(x,u_\varepsilon)\|_{L^1(\Omega)} &= \int_\Omega |G_\varepsilon(x,u_\varepsilon)| \le \\ &\le \int_\Omega d_1|u_\varepsilon|^r + d_2 \le \\ &\le \|d_1\|_{L^{z',1}(\Omega)} \||u_\varepsilon|^r\|_{L^{z,\infty}(\Omega)} + \|d_2\|_{L^1(\Omega)} \\ &\le c\|d_1\|_{L^{z',1}(\Omega)} \||u_\varepsilon|^{p-1}\|_{L^{\frac{N}{N-p},\infty}(\Omega)} + \|d_2\|_{L^1(\Omega)} \le c. \end{aligned}$$

On the other hand, using $T_k(u_\varepsilon)$ as test function in (5.1), we easily obtain that for some M and L, we have

$$\int_\Omega |\nabla T_k(u_\varepsilon)|^p \le Mk + L, \tag{5.3}$$

for every $k > 0$ and every ε.

Such an estimate and the growth condition (3.10) on K_ε allow us to use standard techniques (cf. [BG2], [BMu] and [DMOP]) to say that a subsequence (which we still denote by ε) of the indices ε exists such that

$$\begin{cases} u_\varepsilon \to u & \text{almost everywhere in } \Omega, \\ \nabla u_\varepsilon \to \nabla u & \text{almost everywhere in } \Omega, \\ \nabla T_k(u_\varepsilon) \to \nabla T_k(u) & \text{in } (L^p(\Omega))^N \text{ weakly,} \end{cases} \tag{5.4}$$

for every fixed $k \in \mathbb{N}$, where u is a function which is measurable on Ω, almost everywhere finite and such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k \in \mathbb{N}$, with a gradient ∇u as introduced in (2.19).

By (5.3) and by Fatou lemma, we deduce that

$$\int_\Omega |\nabla T_k(u)|^p \leq Mk + L,$$

and Lemma 3.1 gives

$$|u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega) \text{ and } |\nabla u|^{p-1} \in L^{\frac{N}{N-1},\infty}(\Omega).$$

From (5.4) and the definition (3.8) of H_ε, we deduce that

$$H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to H(x, u, \nabla u) \ \text{ almost everywhere in } \Omega. \tag{5.5}$$

Moreover using the growth condition (3.12) on H_ε, the generalized Hölder inequality (2.5) and Theorem 3.2, with a computation similar to (5.2)

$$H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \quad \text{is equi-integrable.}$$

Therefore Vitali Theorem implies that

$$H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to H(x, u, \nabla u) \text{ in } L^1(\Omega) \text{ strongly.}$$

In a similar way we prove that

$$G_\varepsilon(x, u_\varepsilon) \to G(x, u) \text{ in } L^1(\Omega) \text{ strongly.}$$

Therefore the solution u_ε of (3.17) satisfies

$$\begin{cases} -\mathrm{div}(a(x, u_\varepsilon, \nabla u_\varepsilon) + K_\varepsilon(x, u_\varepsilon)) = f_\varepsilon - \Psi_\varepsilon - \mathrm{div}(g) + \mathrm{div}(F) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus & \text{in } \mathcal{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{5.6}$$

where u_ε satisfies (5.4) and

$$\Psi_\varepsilon = H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) + G_\varepsilon(x, u_\varepsilon) \to H(x, u, \nabla u) + G(x, u) \quad \text{in } L^1(\Omega) \text{ strongly,}$$

where $g \in (L^{p'}(\Omega))^N$, $F \in (L^{p'}(\Omega))^N$ and where f_ε, $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ satisfy (3.4), (3.5) and (3.6).

Since u_ε is a weak solution of (5.6), it is also a renormalized solution of (5.6). Therefore we can apply the stability result in the Section 4, which is an extension of Theorem 3.2 proved in [DMOP] when $K(x,s) = 0$ (see also [MP] and [M]). It allows us to assert that u is a renormalized solution of

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u) + K(x,u)) + H(x,u,\nabla u) + G(x,u) = f - \mathrm{div}(g) & \\ \qquad\qquad +\mu_s^+ - \mu_s^- + \mathrm{div}(F) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

which proves Theorem 2.10. ■

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